

SARD'S APPROXIMATION PROCESSES AND OBLIQUE PROJECTIONS

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ABSTRACT. In this paper three problems arising in approximation theory are studied. These problems have already been studied by Arthur Sard. The main goal of this paper is to use geometrical compatibility theory to extend Sard's results and get characterizations of the sets of solutions.

1. INTRODUCTION

In 1950, Arthur Sard [17] proposed an operator theoretic approach to study some problems arising in approximation theory. In his terminology, a process is an operator T on a Hilbert space \mathcal{H} , which is used to approximate $x \in \mathcal{H}$ in the sense that if δx is the error, then $T(x + \delta x)$ approximates x . He studied approximation, least squares and curve fitting processes. His definitions, in a somewhat different notation, are the following:

- a. Given a closed subspace \mathcal{S} of \mathcal{H} , δx a random variable with values in \mathcal{H} , and $E(\cdot)$ the expectation operator (see Section 3 for proper definitions), the operator T is called an *approximation process over \mathcal{S}* if
 - I. $E(T(x + \delta x)) = x$, for every $x \in \mathcal{S}$,
 - II. $E\|T\delta x\|^2 \leq E\|U\delta x\|^2$ for every operator U satisfying condition I.
- b. Let A be a positive (semidefinite) operator in \mathcal{H} and let \mathcal{S} be a closed subspace of \mathcal{H} . The operator T is a *weighted least square process on \mathcal{S}* , if T has range in \mathcal{S} and for every $y \in \mathcal{H}$, $\|A^{1/2}(y - Ty)\| \leq \|A^{1/2}(y - s)\|$ for every $s \in \mathcal{S}$.
- c. Let δx be a random variable with values in \mathcal{H} , and $E(\cdot)$ the expectation operator. Given a basis of \mathcal{S} , $\{v_n\}_{n=1, \dots, N}$, and a set of vectors $\{w_n\}_{n=1, \dots, N}$, the operator $T = \sum_{n=1}^N \langle \cdot, w_n \rangle v_n$ is a *curve fitting process* if
 - I. $E(\sum_{n \in \mathcal{I}} \langle w_n, x + \delta x \rangle v_n) = x$, for every $x \in \mathcal{S}$,
 - II. $E(\sum_{n \in \mathcal{I}} |\langle w_n, \delta x \rangle|^2) \leq E(\sum_{n \in \mathcal{I}} |\langle u_n, \delta x \rangle|^2)$, for every $\{u_n\}_{n=1, \dots, N}$ satisfying condition I.

Recall that a positive semidefinite operator A and a closed subspace \mathcal{S} are called compatible if there exists a (bounded linear) projection Q with range \mathcal{S} which is self-adjoint with respect to the sesquilinear form $\langle \xi, \eta \rangle_A = \langle A\xi, \eta \rangle$ for every $\xi, \eta \in \mathcal{H}$, i.e., if $AQ = Q^*A$. Denote $\mathcal{P}(A, \mathcal{S})$ the set

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of all projections Q such that $AQ = Q^*A$ and $R(Q) = \mathcal{S}$. If $\mathcal{P}(A, \mathcal{S})$ is not empty, it contains a distinguished element, denoted $P_{A, \mathcal{S}}$, which has nullspace $A(\mathcal{S})^\perp \ominus (\mathcal{S} \cap N(A))$ (here $\mathcal{M} \ominus \mathcal{N} = \mathcal{M} \cap (\mathcal{M} \cap \mathcal{N})^\perp$). Analogously, denote $\mathcal{P}_*(A, \mathcal{S})$ the set of all projections that $AQ^* = QA$ and $R(Q) = \mathcal{S}$. It turns out that this notion allows a geometrical approach to the technical results of Sard. The main goal of this paper is to characterize the sets of processes defined by Sard. In these characterizations, compatibility plays a central role. The notion of compatibility is related to Schur complements [1, 8, 9, 14], Ando complementability [2, 16], abstract splines in Hilbert spaces [6, 10, 15], weighted pseudo inverses [7], frame theory [5], signal processing [12, 13], sampling theory [4, 18], and so on.

The main results of the paper are the following:

- a. The set of A -approximation processes over \mathcal{S} is not empty if and only if A is compatible with \mathcal{S}^\perp ; moreover, it coincides with $\mathcal{P}_*(A, \mathcal{S}^\perp) + L(N(A) \cap \mathcal{S}, \mathcal{S})$.
- b. There exists an A -weighted least square process on \mathcal{S} if and only if A and \mathcal{S} are compatible; in this case, these processes are operators of the form $P_{A, \mathcal{S}} + T$, with $T \in L(\mathcal{H}, \mathcal{S} \cap N(A))$.
- c. Given positive operators $A, B \in L(\mathcal{H})^+$ and a closed subspace \mathcal{S} of \mathcal{H} there exists a B -approximation process which is also an A -weighted least square process on \mathcal{S} if and only if (A, \mathcal{S}) is compatible and the Dixmier angle between $N(A) \cap \mathcal{S}$ and $B(\mathcal{S}^\perp)$ is positive.
- d. Given a positive trace class operator $A \in L(\mathcal{H})$ and a closed subspace \mathcal{S} of \mathcal{H} , there exists an A -curve fitting process on \mathcal{S} if and only if (A, \mathcal{S}^\perp) is compatible. Furthermore we give a characterization of these processes.

2. PRELIMINARIES

Along this work \mathcal{H} denotes a (complex, separable) Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Given two Hilbert spaces \mathcal{H} and \mathcal{K} , $L(\mathcal{H}, \mathcal{K})$ is the space of bounded linear operators from \mathcal{H} into \mathcal{K} and $L(\mathcal{H}) = L(\mathcal{H}, \mathcal{H})$. If $T \in L(\mathcal{H})$ then T^* denotes the adjoint operator of T , $R(T)$ stands for the range of T and $N(T)$ for its nullspace. If \mathcal{S} is a closed subspace of \mathcal{H} and \mathcal{T} is a closed subspace of \mathcal{K} , then $L(\mathcal{S}, \mathcal{T})$ will be identified with the subspace of $L(\mathcal{H}, \mathcal{K})$ consisting of all $T \in L(\mathcal{H}, \mathcal{K})$ such that $R(T) \subseteq \mathcal{T}$ and $\mathcal{S}^\perp \subseteq N(T)$.

Let $L(\mathcal{H})^+$ be the cone of (semidefinite) positive operators of $L(\mathcal{H})$ and denote by \mathcal{Q} the set of projections of $L(\mathcal{H})$, i.e., $\mathcal{Q} = \{Q \in L(\mathcal{H}) : Q^2 = Q\}$.

If \mathcal{S} and \mathcal{T} are two (closed) subspaces of \mathcal{H} , denote by $\mathcal{S} \dot{+} \mathcal{T}$ the direct sum of \mathcal{S} and \mathcal{T} , $\mathcal{S} \oplus \mathcal{T}$ the (direct) orthogonal sum of them and $\mathcal{S} \ominus \mathcal{T} = \mathcal{S} \cap (\mathcal{S} \cap \mathcal{T})^\perp$. If $\mathcal{H} = \mathcal{S} \dot{+} \mathcal{T}$, the oblique projection $P_{\mathcal{S} // \mathcal{T}}$ onto \mathcal{S} along \mathcal{T} is the projection with $R(P_{\mathcal{S} // \mathcal{T}}) = \mathcal{S}$ and $N(P_{\mathcal{S} // \mathcal{T}}) = \mathcal{T}$. In particular, $P_{\mathcal{S}} = P_{\mathcal{S} // \mathcal{S}^\perp}$ is the orthogonal projection onto \mathcal{S} .

Given two subspaces \mathcal{S}, \mathcal{T} , the cosine of the *Friedrichs angle* $\theta(\mathcal{S}, \mathcal{T}) \in [0, \pi/2]$ between them is defined by

$$c(\mathcal{S}, \mathcal{T}) = \sup\{|\langle x, y \rangle| : x \in \mathcal{S} \ominus \mathcal{T}, \|x\| < 1, y \in \mathcal{T} \ominus \mathcal{S}, \|y\| < 1\}.$$

The following conditions are equivalent:

- (1) $c(\mathcal{S}, \mathcal{T}) < 1$;
- (2) $\mathcal{S} + \mathcal{T}$ is closed;
- (3) $\mathcal{S}^\perp + \mathcal{T}^\perp$ is closed;
- (4) $c(\mathcal{S}^\perp, \mathcal{T}^\perp) < 1$.

The *Dixmier angle* between \mathcal{S} and \mathcal{T} is the angle in $[0, \pi/2]$ whose cosine is defined by

$$c_0(\mathcal{S}, \mathcal{T}) = \sup\{|\langle x, y \rangle| : x \in \mathcal{S}, \|x\| < 1, y \in \mathcal{T}, \|y\| < 1\}.$$

Observe that, in general $c(\mathcal{S}, \mathcal{T}) \leq c_0(\mathcal{S}, \mathcal{T})$ and if $\mathcal{S} \cap \mathcal{T} = \{0\}$ then the equality holds. Notice that, if $c_0(\mathcal{S}, \mathcal{T}) < 1$ then $\mathcal{S} \cap \mathcal{T} = \{0\}$.

Given $A \in L(\mathcal{H})^+$ consider the (bounded) sesquilinear form in $\mathcal{H} \times \mathcal{H}$ defined by

$$\langle x, y \rangle_A = \langle Ax, y \rangle, \quad \text{for } x, y \in \mathcal{H},$$

and the corresponding seminorm $\|x\|_A^2 = \langle x, x \rangle_A$.

If \mathcal{S} is a closed subspace of \mathcal{H} and $A \in L(\mathcal{H})^+$, the *A-orthogonal subspace to \mathcal{S}* is given by

$$\mathcal{S}^{\perp A} := \{x \in \mathcal{H} : \langle x, s \rangle_A = 0 \text{ for every } s \in \mathcal{S}\}.$$

It holds that $\mathcal{S}^{\perp A} = A^{-1}(\mathcal{S}^\perp) = A(\mathcal{S})^\perp$.

An operator $T \in L(\mathcal{H})$ is *A-selfadjoint* if $\langle Tx, y \rangle_A = \langle x, Ty \rangle_A$ for every $x, y \in \mathcal{H}$. It is easy to see that T satisfies this condition if and only if $AT = T^*A$.

Definition 2.1. Let $A \in L(\mathcal{H})^+$ and \mathcal{S} be a closed subspace of \mathcal{H} . The pair (A, \mathcal{S}) is *compatible* if there exists an *A-selfadjoint* projection with range \mathcal{S} , i.e. if the set

$$\mathcal{P}(A, \mathcal{S}) = \{Q \in \mathcal{Q} : R(Q) = \mathcal{S}, AQ = Q^*A\}$$

is not empty.

Observe that a projection Q is *A-selfadjoint* if and only if its nullspace satisfies the inclusion $N(Q) \subseteq R(Q)^{\perp A}$. Then, it easily follows that (A, \mathcal{S}) is compatible if and only if

$$(2.1) \quad \mathcal{H} = \mathcal{S} + A^{-1}(\mathcal{S}^\perp).$$

Given a compatible pair (A, \mathcal{S}) , let $\mathcal{N} = \mathcal{S} \cap A(\mathcal{S})^\perp$. It is easy to see that $\mathcal{N} = \mathcal{S} \cap N(A)$. The decomposition $\mathcal{H} = \mathcal{S} \dot{+} (A(\mathcal{S})^\perp \ominus \mathcal{N})$ defines the oblique projection

$$(2.2) \quad P_{A, \mathcal{S}} := P_{\mathcal{S} // A(\mathcal{S})^\perp \ominus \mathcal{N}}.$$

Since $R(P_{A, \mathcal{S}}) = \mathcal{S}$ and $N(P_{A, \mathcal{S}}) \subseteq A(\mathcal{S})^\perp$ it follows that $P_{A, \mathcal{S}} \in \mathcal{P}(A, \mathcal{S})$.

The set $\mathcal{P}(A, \mathcal{S})$ is an affine manifold that, for a given $P \in \mathcal{P}(A, \mathcal{S})$, can be parametrized as

$$(2.3) \quad \mathcal{P}(A, \mathcal{S}) = P + L(\mathcal{S}^\perp, \mathcal{N}).$$

For a proof of these facts see [8, Theorem 3.5].

The following list contains examples of compatible and non compatible pairs

- Example 2.2.** Suppose that $A \in L(\mathcal{H})^+$ and \mathcal{S} is a closed subspace of \mathcal{H} ,
- If A has closed range, the pair (A, \mathcal{S}) is compatible if and only if $N(A) + \mathcal{S}$ is closed, or equivalently $c(N(A), \mathcal{S}) < 1$;
 - If $R(P_{\mathcal{S}}AP_{\mathcal{S}})$ is closed then the pair (A, \mathcal{S}) is compatible;
 - In particular, if $P_{\mathcal{S}}AP_{\mathcal{S}}$ is invertible in $L(\mathcal{S}, \mathcal{S})$ then the pair (A, \mathcal{S}) is compatible, moreover $\mathcal{P}(A, \mathcal{S}) = \{P_{A, \mathcal{S}}\}$. See [3] for a proof of this fact .
 - If \mathcal{S} has finite dimension then (A, \mathcal{S}) is compatible. This fact follows directly from item a.;
 - $c_0(\mathcal{S}^{\perp}, \overline{A(\mathcal{S})}) < 1$ if and only if the pair (A, \mathcal{S}) is compatible.

An interesting example, where the pair (A, \mathcal{S}) is not compatible, can be found in [17, example 12].

The next two concepts were introduced by A. Sard in [17] in order to establish necessary and sufficient conditions for the existence of operators such as approximation, least squares and curve fitting processes.

Definition 2.3. Let \mathcal{T} be a closed subspace of \mathcal{H} . An operator $C \in L(\mathcal{H})$ is *proper on \mathcal{T}* if there exists $D \in L(\mathcal{H})$ such that $DP_{\mathcal{T}}CP_{\mathcal{T}} = P_{\mathcal{T}}$.

Definition 2.4. Let $A \in L(\mathcal{H})^+$ and \mathcal{T} be a closed subspace of \mathcal{H} . An operator $C \in L(\mathcal{H})$ is a *companion of A relative to \mathcal{T}* if it is proper on \mathcal{T} and $R(AC^*P_{\mathcal{T}}) \subseteq P_{\mathcal{T}}$.

There is a close relationship between the compatibility of the pair (A, \mathcal{S}) and the existence of companions of A relative to \mathcal{S}^{\perp} ; in fact, these two concepts are equivalent.

Proposition 2.5. *Let $A \in L(\mathcal{H})^+$ and \mathcal{S} be a closed subspace of \mathcal{H} . There exists a companion of A relative to \mathcal{S}^{\perp} if and only if the pair (A, \mathcal{S}) is compatible.*

Proof. Suppose that the pair (A, \mathcal{S}) is compatible and let $E = I - Q^*$, for $Q \in \mathcal{P}(A, \mathcal{S})$; obviously E is a projection and $R(E) = N(Q^*) = R(Q)^{\perp} = \mathcal{S}^{\perp}$. Then $EP_{\mathcal{S}^{\perp}}EP_{\mathcal{S}^{\perp}} = P_{\mathcal{S}^{\perp}}$, i.e., E is proper on \mathcal{S}^{\perp} . Furthermore,

$$AE^* = A(I - Q) = A - Q^*A = EA,$$

hence $R(AE^*) \subseteq \mathcal{S}^{\perp}$, which implies that E is a companion of A relative to \mathcal{S}^{\perp} .

Conversely, suppose that $C \in L(\mathcal{H})$ is a companion of A relative to \mathcal{S}^{\perp} ; a fortiori, C is proper on \mathcal{S}^{\perp} . Let $D \in L(\mathcal{H})$ such that $DP_{\mathcal{S}^{\perp}}CP_{\mathcal{S}^{\perp}} = P_{\mathcal{S}^{\perp}}$ and consider $Q = (I - P_{\mathcal{S}^{\perp}}DP_{\mathcal{S}^{\perp}}C)^*$. It is easy to see that $E = P_{\mathcal{S}^{\perp}}DP_{\mathcal{S}^{\perp}}C$ is a projection with range \mathcal{S}^{\perp} . Then Q is a projection with $R(Q) = \mathcal{S}$. Furthermore, since $R(AC^*P_{\mathcal{S}^{\perp}}) \subseteq \mathcal{S}^{\perp}$, then $EAE^* = AE^*$ and, since EAE^* is selfadjoint, it holds $AE^* = EA$, which implies that $AQ = Q^*A$, i.e., $Q \in \mathcal{P}(A, \mathcal{S})$. \square

In particular, the above proof shows that if Q is an A -selfadjoint projection with range \mathcal{S}^\perp then $E = I - Q^*$ is a companion of A relative to \mathcal{S} . However, not every companion is necessarily a projection (see Theorem 3.4, below).

There also exists a link between the compatibility of (A, \mathcal{S}) and the condition of A being proper on \mathcal{S} , as show the following results.

Lemma 2.6. *Let $A \in L(\mathcal{H})^+$ and \mathcal{S} be a closed subspace of \mathcal{H} . Then, A is proper on \mathcal{S} if and only if $R(P_{\mathcal{S}}AP_{\mathcal{S}}) = \mathcal{S}$.*

Proof. If A is proper on \mathcal{S} , then there exists $D \in L(\mathcal{H})$ such that $DP_{\mathcal{S}}AP_{\mathcal{S}} = P_{\mathcal{S}}$. Let $B = P_{\mathcal{S}}AP_{\mathcal{S}}$; then $B \in L(\mathcal{H})^+$ and $R(B) \subseteq \mathcal{S}$. From $BD^* = P_{\mathcal{S}}$, it follows that $R(B) = \mathcal{S}$.

Conversely, if $R(B) = \mathcal{S}$, then $B^\dagger B = P_{\mathcal{S}}$, where B^\dagger denotes the Moore-Penrose pseudo inverse of B . \square

Proposition 2.7. *Let $A \in L(\mathcal{H})^+$ and \mathcal{S} be a closed subspace of \mathcal{H} . Then A is proper on \mathcal{S} if and only if $\mathcal{N} = \{0\}$ and $R(P_{\mathcal{S}}AP_{\mathcal{S}})$ is closed. In this case (A, \mathcal{S}) is compatible, $\mathcal{P}(A, \mathcal{S}) = \{P_{A, \mathcal{S}}\}$ and $P_{A, \mathcal{S}} = P_{\mathcal{S}}(I + DP_{\mathcal{S}}AP_{\mathcal{S}^\perp})$, where $D \in L(\mathcal{H})$ satisfies $DP_{\mathcal{S}}AP_{\mathcal{S}} = P_{\mathcal{S}}$.*

Proof. Let $B = P_{\mathcal{S}}AP_{\mathcal{S}}$, notice that $N(B) = \mathcal{N} \oplus \mathcal{S}^\perp$. From Lemma 2.6, if A is proper on \mathcal{S} then $R(B) = \mathcal{S}$, therefore it is closed. Also, $N(B) = \mathcal{S}^\perp$, so that $\mathcal{N} = \{0\}$.

Conversely, if $R(B)$ is closed and $\mathcal{N} = \{0\}$, then $N(B) = \mathcal{S}^\perp$ and $\overline{R(B)} = R(B) = \mathcal{S}$. In this case, from [8, Remark 2.12 (item 2)], it follows that (A, \mathcal{S}) is compatible. Observe that if $DB = P_{\mathcal{S}}$ then $P_{\mathcal{S}}DP_{\mathcal{S}}B = P_{\mathcal{S}}$ and $R(P_{\mathcal{S}}DP_{\mathcal{S}}) = \mathcal{S}$. Let $C = P_{\mathcal{S}}DP_{\mathcal{S}}$, it follows that $BC^* = P_{\mathcal{S}}$ so that $P_{\mathcal{S}}C^* = B^\dagger P_{\mathcal{S}} = B^\dagger$, or $C^* = B^\dagger = C$. From [8, Remark 2.12 (item 1)], we get $P_{A, \mathcal{S}} = P_{\mathcal{S}} + B^\dagger P_{\mathcal{S}}AP_{\mathcal{S}^\perp} = P_{\mathcal{S}} + P_{\mathcal{S}}DP_{\mathcal{S}}AP_{\mathcal{S}^\perp}$. \square

Notice that, if A has closed range, then A is proper on \mathcal{S} if and only if the pair (A, \mathcal{S}) is compatible and $\mathcal{N} = \{0\}$, because the compression $P_{\mathcal{S}}AP_{\mathcal{S}}$ has closed range (see [8, Theorem 6.2]).

3. APPROXIMATION PROCESSES

Let μ be a Lebesgue-Stieltjes measure on \mathbb{R} and let \mathcal{H} be the Hilbert space $L^2(\mu)$. Suppose that (Ω, \mathcal{F}, P) is a probability space, if $z : \Omega \rightarrow \mathbb{R}$ is P -measurable then the *expectation* of z is $E(z) = \int_{\Omega} z(\omega) dP(\omega)$.

Let $\delta x : \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ be a $\mu \times P$ -measurable function such that:

- (1) for almost every $t \in \mathbb{R}$, $E(\delta x(t, \cdot)) = 0$,
- (2) for almost every $\omega \in \Omega$, $\delta x(\cdot, \omega) \in \mathcal{H}$,
- (3) $E(\|\delta x\|^2) = \int_{\Omega} \int_{\mathbb{R}} |\delta x(\omega, t)|^2 d\mu(t) dP(\omega) < \infty$.

The *variance operator* $A \in L(\mathcal{H})^+$ of δx is defined by

$$Ax = E(\langle x, \delta x \rangle \delta x) = \int_{\Omega} \delta x(\omega, \cdot) \int_{\mathbb{R}} \delta x(\omega, t) x(t) d\mu(t) dP(\omega),$$

for every $x \in \mathcal{H}$. As it is shown in [17, Lemma 2], the variance operator A is a trace class operator.

In signal processing applications, x is a (finite energy) signal which has to be estimated and δx a noise measurement. Given the measurement $x + \delta x$, we have to recover the signal x by means of a filter (i.e., an operator) $T \in L(\mathcal{H})$. In general, the reconstructed signal $T(x + \delta x)$ does not coincide with the signal x . It may happen that, at least in a suitable set of signals, the expected value of the reconstructed signal $E(T(x + \delta x))$ coincides with x . If there exist many operators that satisfy our requirement, we add a restriction, for instance that the incidence of the noise in the reconstructed signal be minimized. This problem has been studied in [17] and motivates the definition of approximation processes.

Definition 3.1. Given a closed subspace \mathcal{T} of \mathcal{H} and δx (with the above assumptions), let $\mathcal{U} = \{T \in L(\mathcal{H}) : E(T(x + \delta x)) = x, \text{ for every } x \in \mathcal{T}\}$. Then, $T \in \mathcal{U}$ is called an *approximation process over \mathcal{T}* if $E\|T\delta x\|^2 \leq E\|U\delta x\|^2$ for every $U \in \mathcal{U}$.

Since $E(T(x + \delta x)) = \int_{\Omega} Tx + T\delta x(\omega, \cdot) dP(\omega) = Tx$, because $E(T\delta x) = 0$ (see [17, Lemma 3]), then every element of \mathcal{U} satisfies $Tx = x$ for every $x \in \mathcal{T}$, or, which is the same, $\mathcal{T} \subseteq N(I - T)$. The quantity $E(\|T\delta x\|^2)$ is related with the variance of δx . In fact, if $A \in L(\mathcal{H})^+$ is the variance operator of δx , then $E\|T\delta x\|^2 = \text{Tr}(TAT^*)$ (see [17, Lemma 3]). The following theorem, due to Sard, gives a characterization of an approximation process.

Theorem [17, Theorem 1] Let $A \in L(\mathcal{H})^+$ be the variance operator of certain stochastic process δx and \mathcal{S} be a closed subspace of \mathcal{H} . Then $T \in L(\mathcal{H})$ is an approximation process over \mathcal{S}^\perp with variance A if and only if $R(AT^*) \subseteq \mathcal{S}^\perp \subseteq N(I - T)$. Moreover, if $N(A) \cap \mathcal{S} = \{0\}$, then there exists a unique approximation process T .

Based on this theorem we state the following definition of a generalized approximation process, in which we do not constrain $A \in L(\mathcal{H})^+$ to be a trace class operator.

Definition 3.2. Given $A \in L(\mathcal{H})^+$ and \mathcal{S} a closed subspace of \mathcal{H} . $T \in L(\mathcal{H})$ is an *A-approximation process over \mathcal{S}^\perp* , if $R(AT^*) \subseteq \mathcal{S}^\perp \subseteq N(I - T)$.

Sard proved ([17], Theorem 2) that a necessary and sufficient condition for the existence of an *A-approximation process over \mathcal{S}^\perp* is that there exists a companion of A relative to \mathcal{S}^\perp , or equivalently, by Proposition 2.5, that the pair (A, \mathcal{S}) is compatible. This suggests a relationship between the set of *A-selfadjoint projections with range \mathcal{S}* and the *A-approximation processes over \mathcal{S}* .

For $A \in L(\mathcal{H})^+$ and \mathcal{S}^\perp a closed subspace, let $\mathcal{P}_*(A, \mathcal{S}^\perp) = \{Q \in \mathcal{Q} : R(Q) = \mathcal{S}^\perp, AQ^* = QA\}$.

Remark 3.3. Observe that $\mathcal{P}_*(A, \mathcal{S}^\perp)$ is an affine manifold, eventually void: in fact, $Q \in \mathcal{P}_*(A, \mathcal{S}^\perp)$ if and only if $(I - Q^*) \in \mathcal{P}(A, \mathcal{S})$. Then by equation 2.3, $\mathcal{P}_*(A, \mathcal{S}^\perp) = (I - P_{A, \mathcal{S}}^*) + L(N(A) \cap \mathcal{S}, \mathcal{S}^\perp)$. This manifold is contained in the set of *A-approximation processes on \mathcal{S}^\perp* , as shows the following theorem.

Theorem 3.4. *Let $A \in L(\mathcal{H})^+$ and \mathcal{S} be a closed subspace of \mathcal{H} . Let \mathcal{A} be the set of A -approximation processes over \mathcal{S}^\perp . Then \mathcal{A} is not empty if and only if the pair (A, \mathcal{S}) is compatible. In this case,*

$$\mathcal{A} = \mathcal{P}_*(A, \mathcal{S}^\perp) + L(\mathcal{N}, \mathcal{S}).$$

Proof. Suppose that $T \in L(\mathcal{H})$ is an A -approximation process over \mathcal{S}^\perp ; then $R(AT^*) \subseteq N(I - T)$ so that $AT^* = TAT^* = TA$. Therefore T is A -selfadjoint. From $R(AT^*) \subseteq \mathcal{S}^\perp$, we conclude that $P_{\mathcal{S}}AT^* = 0 = TAP_{\mathcal{S}}$. From $\mathcal{S}^\perp \subseteq N(I - T)$, it follows that $T = P_{\mathcal{S}^\perp} + TP_{\mathcal{S}}$, and then $I - T^* = P_{\mathcal{S}} - P_{\mathcal{S}}T^* = P_{\mathcal{S}} - P_{\mathcal{S}}T^*P_{\mathcal{S}} - P_{\mathcal{S}}T^*P_{\mathcal{S}^\perp}$. Let $Q = P_{\mathcal{S}} - P_{\mathcal{S}}T^*P_{\mathcal{S}^\perp}$; it is easy to check that Q is a projection with range \mathcal{S} . Also, from $AT^* = AP_{\mathcal{S}^\perp} + AP_{\mathcal{S}}T^*$, it follows that $AT^*P_{\mathcal{S}} = TAP_{\mathcal{S}} = 0 = AP_{\mathcal{S}}T^*P_{\mathcal{S}}$. Finally, observing that $A(I - T^*) = (I - T)A$, and that $I - T^* = Q - P_{\mathcal{S}}T^*P_{\mathcal{S}}$, we get that $Q \in \mathcal{P}(A, \mathcal{S})$. Then $I - Q + W = T^*$, with $W = P_{\mathcal{S}}T^*P_{\mathcal{S}} \in L(\mathcal{S}, \mathcal{N})$ and $Q \in \mathcal{P}(A, \mathcal{S})$.

Conversely, if $T^* = I - Q + W$, with $Q \in \mathcal{P}(A, \mathcal{S})$ and $W \in L(\mathcal{S}, \mathcal{N})$, then it is easy to see that T^* is A -selfadjoint. Moreover, $R(AT^*) = R(A(I - Q)) = R((I - Q^*)A) \subseteq R(I - Q^*) = \mathcal{S}^\perp$ and $(I - T)P_{\mathcal{S}^\perp} = (Q^* - W^*)P_{\mathcal{S}^\perp} = 0$, so that $\mathcal{S}^\perp \subseteq N(I - T)$. \square

An alternative characterization of \mathcal{A} is given by

$$\mathcal{A} = \{T \in L(\mathcal{H}) : T = I - P_{A, \mathcal{S}}^* + W, \text{ where } W \in L(\mathcal{N}, \mathcal{H})\}.$$

This follows from the theorem above and equation 2.3.

4. WEIGHTED LEAST SQUARES PROCESSES

Let $A \in L(\mathcal{H})^+$, \mathcal{S} be a closed subspace of \mathcal{H} and $y \in \mathcal{H}$. Any $u \in \mathcal{S}$ such that

$$(4.1) \quad \|y - u\|_A = \min_{x \in \mathcal{S}} \|y - x\|_A,$$

is called a *weighted least squares approximation of y in \mathcal{S} (with weight A)* (hereafter A -WLSA).

In [17], the problem of finding an operator $T \in L(\mathcal{H})$ which assigns to each $y \in \mathcal{H}$ an A -WLSA is studied. Such operator is called a *weighted least square process with weight A (A -WLSP)*.

Definition 4.1. Let $A \in L(\mathcal{H})^+$ and \mathcal{S} be a closed subspace of \mathcal{H} . $T \in L(\mathcal{H}, \mathcal{S})$ is an *A -weighted least square process (A -WLSP)* on \mathcal{S} , if for every $y \in \mathcal{H}$, $\|y - Ty\|_A \leq \|y - s\|_A$ for every $s \in \mathcal{S}$.

If the weight A is proper on \mathcal{S} , Sard proved that there exists a unique A -WLSP and it is an A -selfadjoint projection. Later, the same problem, with different motivations than those of Sard, has been studied in [7] and [10] under compatibility hypothesis; it has been shown that the compatibility of the pair (A, \mathcal{S}) is not only a sufficient but also a necessary condition for the existence of a A -WLSP. We summarize some of these results, more precisely [7, Proposition 4.4] and [10, Theorem 3.2] in the following statement. Notice, however, that we use the notation of the present paper, which is essentially that of Sard.

Theorem 4.2. *Let $A \in L(\mathcal{H})^+$ and \mathcal{S} be a closed subspace of \mathcal{H} . The following conditions hold:*

- (1) *There exists an A -WLSA on \mathcal{S} , for every $y \in \mathcal{H}$, if and only if the pair (A, \mathcal{S}) is compatible. In such case, it is unique if and only if $\mathcal{N} = \{0\}$.*
- (2) *$u \in \mathcal{S}$ is an A -WLSA of y if and only if $y - u \in A(\mathcal{S})^\perp$.*
- (3) *Every $Q \in \mathcal{P}(A, \mathcal{S})$ is an A -WLSP on \mathcal{S} .*

Corollary 4.3. *There exists an A -WLSA on \mathcal{S} , for every $y \in \mathcal{S}^\perp$, if and only if the pair (A, \mathcal{S}) is compatible.*

Proof. Suppose that, for every $\tilde{y} \in \mathcal{S}^\perp$, there exists an A -WLSA on \mathcal{S} . Given $y \in \mathcal{H} \setminus \mathcal{S}$, let z_0 the A -WLSA of $P_{\mathcal{S}^\perp}y$, then

$$(4.2) \quad \|z_0 - P_{\mathcal{S}^\perp}y\|_A \leq \|z - P_{\mathcal{S}^\perp}y\|_A, \text{ for every } z \in \mathcal{S}.$$

Let $x_0 = z_0 + P_{\mathcal{S}}y$ and $x = z + P_{\mathcal{S}}y$. Then

$$\|y - x_0\|_A \leq \|y - x\|_A.$$

Since equation 4.2 holds for any $z \in \mathcal{S}$, then x is an arbitrary vector in \mathcal{S} , thus x_0 is an A -WLSP for $y \in \mathcal{H} \setminus \mathcal{S}$. Furthermore, if $y \in \mathcal{S}$, then $x_0 = y$ is an A -WLSA on \mathcal{S} , because $A \in L(\mathcal{H})^+$. Then, there exists an A -WLSA for every $y \in \mathcal{H}$, thus, by Theorem 4.2, (A, \mathcal{S}) is compatible.

The converse follows directly from Theorem 4.2. □

By Proposition 2.7, it is clear that the condition of A being proper on \mathcal{S} is a sufficient but not necessary condition for the existence and uniqueness of A -WLSA. Furthermore, item (3) shows that every element in $\mathcal{P}(A, \mathcal{S})$ is an A -WLSP. It should be noticed, however that, an A -WLSP may not be a projection, as shows the next proposition.

Proposition 4.4. *Let $A \in L(\mathcal{H})^+$ and \mathcal{S} be a closed subspace of \mathcal{H} such that the pair (A, \mathcal{S}) is compatible. The operator $W \in L(\mathcal{H})$ is an A -WLSP if and only if $W \in P_{A, \mathcal{S}} + L(\mathcal{H}, \mathcal{N})$.*

Proof. Suppose that $W \in L(\mathcal{H})$ is a A -WLSP. Then, by Theorem 4.2 (item (2)), for every $y \in \mathcal{H}$, it holds $y - Wy \in A(\mathcal{S})^\perp$, i.e.

$$\langle Wy - y, z \rangle_A = 0, \text{ for every } z \in \mathcal{S}.$$

Then,

$$0 = \langle Wy - y, P_{A, \mathcal{S}}x \rangle_A = \langle Wy - P_{A, \mathcal{S}}y, x \rangle_A, \text{ for every } x \in \mathcal{H},$$

because $R(W) \subseteq \mathcal{S}$, and $L = W - P_{A, \mathcal{S}} \in L(\mathcal{H}, \mathcal{N})$.

Conversely, suppose that $W = P_{A, \mathcal{S}} + L$, where $L \in L(\mathcal{H}, \mathcal{N})$; then, by Proposition 4.2 and the comments above, it is easy to see that, given $y \in \mathcal{H}$, $\langle Wy - y, z \rangle_A = \langle P_{A, \mathcal{S}}y - y, z \rangle_A = 0$, for every $z \in \mathcal{S}$. □

An interesting problem, which naturally appears in some signal processing applications, is the following: given a closed subspace \mathcal{S} and $A, B \in L(\mathcal{H})^+$, find a B -approximation process over \mathcal{S} which is also an A -WLSP on \mathcal{S} . In [17], this problem has been studied under the assumption that the weight A

is proper on \mathcal{S} . In this section we study this problem under the assumption that the pair (A, \mathcal{S}) is compatible.

The next result shows that the set of B -approximation processes over \mathcal{S} which are also A -WLSP on \mathcal{S} , is the intersection of the affine manifolds $\mathcal{P}_*(B, \mathcal{S})$ and $\mathcal{P}(A, \mathcal{S})$. Notice that, if T is a B -approximation process and also an A -WLSP on \mathcal{S} , then T is a projection. If $\mathcal{N} = \{0\}$, as a particular case, if A is proper on \mathcal{S} , the problem of finding B -approximation processes which are also A -WLSP on \mathcal{S} , reduces to check if $P_{A, \mathcal{S}} \in \mathcal{P}_*(B, \mathcal{S})$.

Observe that, by Proposition 3.4 and Theorem 4.2, the compatibility of the pairs (A, \mathcal{S}) and (B, \mathcal{S}^\perp) is a necessary condition for the existence of B -approximation processes which are also A -WLSP over \mathcal{S} .

Lemma 4.5. *Let $A, B \in L(\mathcal{H})^+$ and \mathcal{S} be a closed subspace, such that the pairs (A, \mathcal{S}) and (B, \mathcal{S}^\perp) are compatible. Then $T \in L(\mathcal{H})$ is a B -approximation process over \mathcal{S} and also an A -WLSP on \mathcal{S} , if and only if $T \in \mathcal{P}_*(B, \mathcal{S}) \cap \mathcal{P}(A, \mathcal{S})$.*

Proof. If $T \in \mathcal{P}(A, \mathcal{S}) \cap \mathcal{P}_*(B, \mathcal{S})$; then by Propositions 4.4 and 3.4, it follows that T is a A -WLSP and also an approximation process.

Conversely, suppose that T is a B -approximation process over \mathcal{S} . By Proposition 3.4, $T = (I - Q)^* + Z$, where $Q \in \mathcal{P}(B, \mathcal{S}^\perp)$ and $Z \in L(N(B) \cap \mathcal{S}^\perp, \mathcal{S}^\perp)$. If T is an A -WLSP in \mathcal{S} then $R(T) \subseteq \mathcal{S}$, which implies that $\mathcal{S}^\perp \subseteq R(T)^\perp = N(T^*)$. Since $T^* = (I - Q) + Z$ then for every $x \in \mathcal{S}^\perp$, $Zx = 0$, i.e., $T = (I - Q)^* \in \mathcal{P}_*(B, \mathcal{S})$. Also, by Proposition 4.4, $T = P + R$, with $P \in \mathcal{P}(A, \mathcal{S})$ and $R \in L(\mathcal{S}, N(A) \cap \mathcal{S})$. Since T is a projection with range \mathcal{S} , then it is easy to see that $R = 0$, and then $T \in \mathcal{P}(A, \mathcal{S}) \cap \mathcal{P}_*(B, \mathcal{S})$. \square

The next theorem states necessary and sufficient conditions for the non emptiness of $\mathcal{P}(A, \mathcal{S}) \cap \mathcal{P}_*(B, \mathcal{S})$ and hence, for the existence of approximation processes which are also A -WLSP on \mathcal{S} .

Theorem 4.6. *Let $A, B \in L(\mathcal{H})^+$ and \mathcal{S} be a closed subspace of \mathcal{H} . Let $\mathcal{N} = N(A) \cap \mathcal{S}$ and $\mathcal{M} = N(B) \cap \mathcal{S}^\perp$. The following conditions are equivalent:*

- (1) $\mathcal{P}(A, \mathcal{S}) \cap \mathcal{P}_*(B, \mathcal{S}) \neq \emptyset$
- (2) (A, \mathcal{S}) is compatible, $AB(\mathcal{S}^\perp) \subseteq \mathcal{S}^\perp$ and $c_0(\mathcal{N}, B(\mathcal{S}^\perp)) < 1$.
- (3) (B, \mathcal{S}^\perp) is compatible, $BA(\mathcal{S}) \subseteq \mathcal{S}$ and $c_0(\mathcal{M}, A(\mathcal{S})) < 1$.

Proof. (1) \Rightarrow (2): Suppose that $Q = P_{\mathcal{S} // \mathcal{T}} \in \mathcal{P}(A, \mathcal{S}) \cap \mathcal{P}_*(B, \mathcal{S})$. Since $Q \in \mathcal{P}(A, \mathcal{S})$, $\mathcal{T} \subseteq A^{-1}(\mathcal{S}^\perp)$. Analogously, $Q^* = P_{\mathcal{T}^\perp // \mathcal{S}^\perp} \in \mathcal{P}(B, \mathcal{T}^\perp)$, because $Q \in \mathcal{P}_*(B, \mathcal{S})$; therefore it follows that $\mathcal{S}^\perp \subseteq B^{-1}(\mathcal{T})$. Then $B(\mathcal{S}^\perp) \subseteq \mathcal{T} \subseteq A^{-1}(\mathcal{S}^\perp)$, i.e. $AB(\mathcal{S}^\perp) \subseteq \mathcal{S}^\perp$. Furthermore, $B(\mathcal{S}^\perp) \cap \mathcal{N} \subseteq \mathcal{T} \cap \mathcal{N} = \{0\}$, then $c(B(\mathcal{N}, \mathcal{S}^\perp)) = c_0(\mathcal{N}, B(\mathcal{S}^\perp))$ and $c(\mathcal{N}, B(\mathcal{S}^\perp)) < c(\mathcal{N}, \mathcal{T}) < c(\mathcal{S}, \mathcal{T}) < 1$.

(2) \Rightarrow (1): Suppose that $AB(\mathcal{S}^\perp) \subseteq \mathcal{S}^\perp$, $\mathcal{N} \cap B(\mathcal{S}^\perp) = \{0\}$ and $c_0(\mathcal{N}, B(\mathcal{S}^\perp)) < 1$. Then $B(\mathcal{S}^\perp) \dot{+} \mathcal{N}$ is a closed subspace of $A^{-1}(\mathcal{S}^\perp)$. Let \mathcal{W} be a closed subspace such that $B(\mathcal{S}^\perp) \dot{+} \mathcal{N} \dot{+} \mathcal{W} = A^{-1}(\mathcal{S}^\perp)$. Let $\mathcal{T} = B(\mathcal{S}^\perp) \dot{+} \mathcal{W} \subseteq A^{-1}(\mathcal{S}^\perp)$; observe that $\mathcal{H} = \mathcal{S} \dot{+} \mathcal{T}$, and define $Q = P_{\mathcal{S} // \mathcal{T}}$. Since $N(Q) \subseteq A^{-1}(\mathcal{S}^\perp)$, it holds $Q \in \mathcal{P}(A, \mathcal{S})$ (see the Preliminaries).

Furthermore, since $B(\mathcal{S}^\perp) \subseteq \mathcal{T}$, then $N(Q^*) = \mathcal{S}^\perp \subseteq B^{-1}(\mathcal{T})$, i.e., Q^* is a B -selfadjoint projection. Thus $Q \in \mathcal{P}(A, \mathcal{S}) \cap \mathcal{P}_*(B, \mathcal{S})$.

(1) \Leftrightarrow (3) follows analogously. \square

Notice that the compatibility of the pair (A, \mathcal{S}) and the conditions $AB(\mathcal{S}^\perp) \subseteq \mathcal{S}^\perp$ and $c_0(\mathcal{N}, B(\mathcal{S}^\perp)) < 1$, imply the compatibility of the pair (B, \mathcal{S}^\perp) .

If $\mathcal{N} = \{0\}$, the hypotheses which guarantee the existence of B -approximation processes over \mathcal{S} which are also A -WLSP on \mathcal{S} , reduce to $AB(\mathcal{S}^\perp) \subseteq \mathcal{S}^\perp$. Then we recover, under compatibility hypothesis, the result proved in [17, Theorem 4] for an $A \in L(\mathcal{H})^+$ which is proper on \mathcal{S} .

The next result gives alternative conditions for the existence of B -approximation processes which are also A -WLSP on \mathcal{S} . This will be useful to construct such kind of operators. Recall that $P_{\mathcal{T}}$ is the orthogonal projection onto \mathcal{T} .

Lemma 4.7. *Suppose that $A, B \in L(\mathcal{H})^+$ and \mathcal{S} is a closed subspace, such that the pairs (A, \mathcal{S}) and (B, \mathcal{S}^\perp) are compatible. Let $P \in \mathcal{P}(A, \mathcal{S})$, $Q \in \mathcal{P}(B, \mathcal{S}^\perp)$, $\mathcal{N} = N(A) \cap \mathcal{S}$ and $\mathcal{M} = N(B) \cap \mathcal{S}^\perp$. Then, the following conditions are equivalent:*

- (1) $\mathcal{P}(A, \mathcal{S}) \cap \mathcal{P}_*(B, \mathcal{S}) \neq \emptyset$,
- (2) $A(I - P - Q^*)P_{\mathcal{M}^\perp} = 0$,
- (3) $B(I - P^* - Q)P_{\mathcal{N}^\perp} = 0$.

Proof. (1) \Rightarrow (2) : Let $T \in \mathcal{P}(A, \mathcal{S}) \cap \mathcal{P}_*(B, \mathcal{S})$. Then, by Lemma 4.5, T is an A -WLSP over \mathcal{S} and a B -approximation process over \mathcal{S} . By equation 2.3 and Remark 3.3,

$$(4.3) \quad T = P + L_1 = (I - Q^*) + L_2,$$

where $L_1 \in L(\mathcal{S}^\perp, \mathcal{N})$ and $L_2 \in L(\mathcal{M}, \mathcal{S})$.

Then $I - Q^* - P = L_1 - L_2$. Let $Z = I - Q^* - P$; it is easy to see that $R(Z) \subseteq \mathcal{S}$ and $Z(\mathcal{M}^\perp) \subseteq \mathcal{N}$ (because $\mathcal{M}^\perp \subseteq N(L_2)$). Then $A(I - Q^* - P)P_{\mathcal{M}^\perp} = 0$.

(2) \Rightarrow (1) : Suppose that $A(I - Q^* - P)P_{\mathcal{M}^\perp} = 0$. Let $Z = (I - Q^* - P)$. Observe that $Z(\mathcal{S}) = 0$ and $R(Z) \subseteq \mathcal{S}$. Furthermore $Z(\mathcal{M}^\perp) \subseteq \mathcal{N}$. Let $L_1 = ZP_{\mathcal{M}^\perp}$ and $L_2 = ZP_{\mathcal{M}}$.

Then $Z = L_1 + L_2 = I - Q^* - P$, or equivalently, $1 - Q^* - L_2 = P + L_1$. If it holds that $T = 1 - Q^* - L_2 = P + L_1$, then $T \in \mathcal{P}(A, \mathcal{S}) \cap \mathcal{P}_*(B, \mathcal{S})$; in fact $R(L_1) \subseteq \mathcal{N}$ and $L_1(\mathcal{S}) = 0$ (because $\mathcal{S} \subseteq \mathcal{M}^\perp$ and $Z(\mathcal{S}) = 0$) so that $L_1 \in L(\mathcal{S}^\perp, \mathcal{N})$. Also, $\mathcal{M}^\perp \subseteq N(L_2)$ and $R(L_2) \subseteq \mathcal{S}$; then $L_2 \in L(\mathcal{M}, \mathcal{S})$.

Then $(I - Q^*) - L_2 = P + L_1$. Denoting $T = P + L_1 = (I - Q^*) - L_2$, it follows that $T \in (P + L(\mathcal{S}^\perp, \mathcal{N})) \cap ((I - Q^*) + L(\mathcal{S}, \mathcal{M})^*)$.

(1) \Leftrightarrow (3) is analogous to (1) \Leftrightarrow (2). \square

The following theorem gives a parametrization of the set of B -approximation processes which are also A -WLSP on \mathcal{S} , i.e., the set $\mathcal{P}(A, \mathcal{S}) \cap \mathcal{P}_*(B, \mathcal{S})$.

Theorem 4.8. *Let $A, B \in L(\mathcal{H})^+$ and a closed subspace \mathcal{S} such that $\mathcal{P}(A, \mathcal{S}) \cap \mathcal{P}_*(B, \mathcal{S}) \neq \emptyset$. Let $\mathcal{N} = N(A) \cap \mathcal{S}$ and $\mathcal{M} = N(B) \cap \mathcal{S}^\perp$.*

Then

$$\mathcal{P}(A, \mathcal{S}) \cap \mathcal{P}_*(B, \mathcal{S}) = (I - P_{B, \mathcal{S}^\perp}^*)P_{\mathcal{M}^\perp} + P_{A, \mathcal{S}}P_{\mathcal{M}} + L(\mathcal{M}, \mathcal{N}).$$

Proof. If $E = (I - P_{B, \mathcal{S}^\perp}^*)P_{\mathcal{M}^\perp} + P_{A, \mathcal{S}}P_{\mathcal{M}} + C$, for some $C \in L(\mathcal{M}, \mathcal{N})$, then $AE = A((I - P_{B, \mathcal{S}^\perp}^*)P_{\mathcal{M}^\perp} + P_{A, \mathcal{S}}P_{\mathcal{M}}) = AP_{A, \mathcal{S}} - AP_{A, \mathcal{S}}P_{\mathcal{M}^\perp} + AP_{\mathcal{M}^\perp} - AP_{B, \mathcal{S}^\perp}^*P_{\mathcal{M}^\perp} = AP_{A, \mathcal{S}}$, since, by Lemma 4.7, $A(I - P_{A, \mathcal{S}} - P_{B, \mathcal{S}^\perp}^*)P_{\mathcal{M}^\perp} = 0$. Then E is A -selfadjoint. Furthermore, since $\mathcal{S} \subseteq \mathcal{M}^\perp$ and $C = CP_{\mathcal{M}}$, it follows that $E^2 = E$. Moreover, it is easy to see that $EP_{\mathcal{S}} = P_{\mathcal{S}}$, then $E \in \mathcal{P}(A, \mathcal{S})$. Analogously, $E \in \mathcal{P}_*(B, \mathcal{S}^\perp)$. Then E is a B -approximation process and also an A -WLSP on \mathcal{S} .

Conversely, let $Q \in \mathcal{P}(A, \mathcal{S}) \cap \mathcal{P}_*(B, \mathcal{S})$ and $C = Q - (I - P_{B, \mathcal{S}^\perp}^*)P_{\mathcal{M}^\perp} - P_{A, \mathcal{S}}P_{\mathcal{M}}$. It follows that $CP_{\mathcal{M}^\perp} = QP_{\mathcal{M}^\perp} - (I - P_{B, \mathcal{S}^\perp}^*)P_{\mathcal{M}^\perp}$. Since $Q \in \mathcal{P}_*(B, \mathcal{S}^\perp)$, it follows that $Q = (I - P_{B, \mathcal{S}^\perp}^*) + W$, with $W \in L(\mathcal{M}, \mathcal{S})$. Then $QP_{\mathcal{M}^\perp} = (I - P_{B, \mathcal{S}^\perp}^*)P_{\mathcal{M}^\perp}$, so that $CP_{\mathcal{M}^\perp} = 0$. Furthermore, since $Q \in \mathcal{P}(A, \mathcal{S})$, $Q = P_{A, \mathcal{S}} + \tilde{W}$, with $\tilde{W} \in L(\mathcal{S}^\perp, \mathcal{N})$. Then, $C = CP_{\mathcal{M}} = P_{A, \mathcal{S}}P_{\mathcal{M}} - \tilde{W}P_{\mathcal{M}} - P_{A, \mathcal{S}}P_{\mathcal{M}}$, so that $R(C) \subseteq \mathcal{N}$. Then, $Q \in (I - P_{B, \mathcal{S}^\perp}^*)P_{\mathcal{M}^\perp} + P_{A, \mathcal{S}}P_{\mathcal{M}} + L(\mathcal{M}, \mathcal{N})$. \square

5. CURVE FITTING PROCESSES

Suppose that $\mathcal{H} = L^2(\mu)$ and $\delta x = \delta x(t, \omega)$ is a stochastic process, as defined in Section 3. Given a linear independent set $\{v_n\}_{n \in \mathcal{I} = \{1, \dots, N\}}$ that span a (finite dimensional) subspace \mathcal{S} of \mathcal{H} and the variance operator of δx , $A \in L(\mathcal{H})^+$; in [17], Sard studied the problem of finding $\{w_n\}_{n \in \mathcal{I}} \subseteq \mathcal{H}$ such that:

$$(5.1) \quad E\left(\sum_{n \in \mathcal{I}} \langle w_n, x + \delta x \rangle v_n\right) = x, \text{ for every } x \in \mathcal{S},$$

and minimize

$$(5.2) \quad E\left(\sum_{n \in \mathcal{I}} |\langle w_n, \delta x \rangle|^2\right).$$

In [17, Lemma 15], it has been proven that

$$E\left(\sum_{n \in \mathcal{I}} |\langle w_n, \delta x \rangle|^2\right) = \sum_{n \in \mathcal{I}} \langle w_n, w_n \rangle_A.$$

In this section we generalize this problem and we study the existence of solutions under the assumption that $\{v_n\}_{n \in \mathcal{I} \subseteq \mathbb{Z}}$ is a frame for a (possibly infinite dimensional) closed subspace \mathcal{S} of \mathcal{H} . First, we introduce some definitions and results.

Definition 5.1. Let \mathcal{S} be a closed subspace of \mathcal{H} . The set $\mathcal{V} = \{v_n\}_{n \in \mathcal{I}} \subseteq \mathcal{S}$ is a *frame* for \mathcal{S} if there exist constants $\gamma_1, \gamma_2 > 0$ such that

$$(5.3) \quad \gamma_1 \|x\|^2 \leq \sum_{n \in \mathcal{I}} |\langle x, v_n \rangle|^2 \leq \gamma_2 \|x\|^2, \text{ for every } x \in \mathcal{S}.$$

If the set $\mathcal{V} = \{v_n\}_{n \in \mathcal{I}}$ is also linearly independent then it is called a *Riesz basis* of \mathcal{S} .

Let \mathcal{S} be a closed subspace of \mathcal{H} and let $\mathcal{V} = \{v_n\}_{n \in \mathcal{I}}$ be a frame for \mathcal{S} . Let \mathcal{K} be a (separable) Hilbert space and $\mathcal{B} = \{e_n\}_{n \in \mathcal{I}}$ be an orthonormal basis of \mathcal{K} . Since equation (5.3) holds, there exists a unique $F \in L(\mathcal{K}, \mathcal{H})$ such that $F e_n = v_n$, for every $n \in \mathcal{I}$. The triplet $(F, \mathcal{B}, \mathcal{K})$ is called the *synthesis operator* of \mathcal{V} , $F^* \in L(\mathcal{H}, \mathcal{K})$ is called the *analysis operator* of \mathcal{V} , and it is given by $F^* x = \sum_{n \in \mathcal{I}} \langle x, v_n \rangle e_n$. The operator $T = FF^* \in L(\mathcal{H})$ does not depend on the synthesis operator F but only on the frame $\mathcal{V} = \{v_n\}_{n \in \mathcal{I}}$ (see [5]), i.e., if $(F_1, \mathcal{K}_1, \mathcal{B}_1)$ is another synthesis operator for \mathcal{V} then $T = F_1 F_1^* = FF^*$; T is called the *frame operator* of \mathcal{V} .

The restriction of T to the subspace \mathcal{S} is invertible; moreover, from equation 5.3, $\gamma_1 P_{\mathcal{S}} \leq P_{\mathcal{S}} T P_{\mathcal{S}} \leq \gamma_2 P_{\mathcal{S}}$ and then $1/\gamma_1 P_{\mathcal{S}} \leq P_{\mathcal{S}} T^\dagger P_{\mathcal{S}} \leq 1/\gamma_2 P_{\mathcal{S}}$, where T^\dagger denotes the Moore-Penrose inverse of T .

Based on the above definitions, the problem stated in equations 5.1 and 5.2 and solved by Sard, can be rewritten as follows. Let $\mathcal{V} = \{v_n\}_{n \in \mathcal{I}=1, \dots, N}$ be a basis of a closed subspace \mathcal{S} and $(F, \{e_n\}_{n \in \mathcal{I}}, \mathbb{C}^N)$ its synthesis operator (where $\{e_n\}_{n \in \mathcal{I}}$ is the canonical basis in \mathbb{C}^N), find $G_0 \in L(\mathbb{C}^N, \mathcal{H})$ such that

- (1) G_0 satisfies $FG_0^* P_{\mathcal{S}} = P_{\mathcal{S}}$,
- (2) $\sum_n \langle G_0 e_k, G_0 e_k \rangle_A \leq \sum_n \langle G e_k, G e_k \rangle_A$ for every $G \in L(\mathbb{C}^N, \mathcal{H})$ such that $FG^* P_{\mathcal{S}} = P_{\mathcal{S}}$.

Notice that $\sum_n \langle G e_k, G e_k \rangle_A = \text{Tr}(G^* A G)$. Based on this reformulation of the original problem, we give the following definition.

Definition 5.2. Let \mathcal{H} and \mathcal{K} be two (separable) Hilbert spaces, $A \in L(\mathcal{H})^+$ a trace class operator and \mathcal{S} be a closed subspace of \mathcal{H} . Let $\mathcal{V} = \{v_n\}_{n \in \mathcal{I}}$ be a frame for \mathcal{S} and $(F, \mathcal{B}, \mathcal{K})$ be a synthesis operator of \mathcal{V} . Then, FG_0^* is an *A-curve fitting process* on \mathcal{S} , if

$$\text{Tr}(G_0^* A G_0) = \min\{\text{Tr}(G^* A G) : G \in L(\mathcal{K}, \mathcal{H}), FG^* P_{\mathcal{S}} = P_{\mathcal{S}}\}.$$

As we will show later, the problem of finding such $G_0 \in L(\mathcal{K}, \mathcal{H})$, is related to an abstract spline problem. We first characterize the set of (bounded linear) operators satisfying condition (1).

Lemma 5.3. Let $\mathcal{V} = \{v_n\}_{n \in \mathcal{I}}$ be a frame for a closed subspace \mathcal{S} , let $(F, \mathcal{B}, \mathcal{K})$ be a synthesis operator for \mathcal{V} and $T = FF^*$. Then $G \in L(\mathcal{K}, \mathcal{H})$ satisfies $FG^* P_{\mathcal{S}} = P_{\mathcal{S}}$ if and only if $GP_{N(F)^\perp} = T^\dagger F + L(N(F)^\perp, \mathcal{S}^\perp)$.

Proof. Since $R(T) = R(F) = \mathcal{S}$, it follows that $P_{\mathcal{S}} = TT^\dagger$. Suppose that $G \in L(\mathcal{K}, \mathcal{H})$ satisfies $FG^* P_{\mathcal{S}} = P_{\mathcal{S}}$, then $F(F^* T^\dagger - G^*)F = 0$. Since $R(F^*)$ is closed, it follows that $F^*(T^\dagger F - G)P_{N(F)^\perp} = 0$, then $R((T^\dagger F - G)P_{N(F)^\perp}) \subseteq N(F^*) = \mathcal{S}^\perp$. Let $W = GP_{N(F)^\perp} - T^\dagger F P_{N(F)^\perp} = GP_{N(F)^\perp} - T^\dagger F$, then $R(W) \subseteq \mathcal{S}^\perp$ and $N(F) \subseteq N(W)$, thus $GP_{N(F)^\perp} \in T^\dagger F + L(N(F)^\perp, \mathcal{S}^\perp)$.

Conversely suppose that $GP_{N(F)^\perp} = T^\dagger F + W$ with $W \in L(N(F)^\perp, \mathcal{S}^\perp)$, then $F^*(T^\dagger F - G)P_{N(F)^\perp} = 0$ which implies that $P_{N(F)^\perp}(G^* - F^* T^\dagger)F = 0$, then $R((G^* - F^* T^\dagger)F) \subseteq N(F)$, i.e., $F(G^* - F^* T^\dagger)F = 0$, so that, $FG^* P_{\mathcal{S}} = P_{\mathcal{S}}$. \square

If $\mathcal{V} = \{v_n\}_{n \in \mathcal{I}}$ is a Riesz basis of \mathcal{S} , then $N(F) = 0$ and Lemma 5.3 asserts that $FG^*P_{\mathcal{S}} = P_{\mathcal{S}}$ if and only if $G = T^\dagger F + L(\mathcal{K}, \mathcal{S}^\perp)$.

Given two Hilbert spaces \mathcal{H} and \mathcal{K} , $C \in L(\mathcal{H}, \mathcal{K})$, \mathcal{M} a closed subspace of \mathcal{H} and $\xi \in \mathcal{H}$, an *abstract spline interpolant* to ξ is any element in the set

$$sp(C, \mathcal{M}, \xi) = \{\eta \in \xi + \mathcal{M} : \|C\eta\| = \min_{\sigma \in \mathcal{M}} \|C(\xi + \sigma)\|\}.$$

See [6, 11, 15].

The following result [10, Theorem 3.2] establishes the relation between compatibility and the existence of abstract spline interpolants.

Theorem 5.4. *Let $C \in L(\mathcal{H}, \mathcal{K})$ and \mathcal{M} be a closed subspace of \mathcal{H} . The set $sp(C, \mathcal{M}, \xi)$ is not empty, for every $\xi \in \mathcal{H}$, if and only if the pair (C^*C, \mathcal{M}) is compatible. Moreover, in this case, $sp(C, \mathcal{M}, \xi) = \{(I - Q)\xi : Q \in \mathcal{P}(C^*C, \mathcal{M})\}$.*

Based on this theorem, we can give conditions for the existence of A -curve fitting processes.

Theorem 5.5. *Let $A \in L(\mathcal{H})^+$ be a trace class operator, \mathcal{S} be a closed subspace of \mathcal{H} , $\mathcal{V} = \{v_n\}_{n \in \mathcal{I}}$ be a frame for \mathcal{S} , $(F, \mathcal{B} = \{e_n\}_{n \in \mathcal{I}}, \mathcal{K})$ the synthesis operator of \mathcal{V} and $T = FF^*$. Then, there exists an A -curve fitting process on \mathcal{S} if and only if the pair (A, \mathcal{S}^\perp) is compatible. Moreover, in this case, given $Q \in \mathcal{P}(A, \mathcal{S}^\perp)$, $G_0 = (I - Q)T^\dagger F$ is such that FG_0 is an A -curve fitting process on \mathcal{S} .*

Proof. Let $\mathcal{I}_1, \mathcal{I}_2 \subseteq \mathcal{I}$, such that $\mathcal{I}_1 \cup \mathcal{I}_2 = \mathcal{I}$, $\{\varepsilon_n\}_{n \in \mathcal{I}_1}$ is an orthonormal basis of $N(F)$ and $\{\varepsilon_n\}_{n \in \mathcal{I}_2}$ is an orthonormal basis of $N(F)^\perp$.

Let $Q \in \mathcal{P}(A, \mathcal{S}^\perp)$, then applying Lemma 5.3 it is easy to see that $G_0 = (I - Q)T^\dagger F \in L(\mathcal{K}, \mathcal{H})$ satisfies $FG_0^*P_{\mathcal{S}} = P_{\mathcal{S}}$.

Suppose that $G \in L(\mathcal{K}, \mathcal{H})$ satisfies $FG^*P_{\mathcal{S}} = P_{\mathcal{S}}$ then, by Lemma 5.3,

$$GP_{N(F)^\perp}\varepsilon_n \in T^\dagger F\varepsilon_n + \mathcal{S}^\perp,$$

and by Theorem 5.4, $\|A^{1/2}(I - Q)T^\dagger F\varepsilon_n\| \leq \|A^{1/2}h\|$ for every $h \in T^\dagger F\varepsilon_n + \mathcal{S}^\perp$, then

$$Tr(G_0^*AG_0) = \sum_{n \in \mathcal{I}} \|G_0\varepsilon_n\|_A^2 = \sum_{n \in \mathcal{I}} \|(I - Q)T^\dagger F\varepsilon_n\|_A^2 = \sum_{n \in \mathcal{I}_2} \|(I - Q)T^\dagger F\varepsilon_n\|_A^2 \leq$$

$$\sum_{n \in \mathcal{I}_2} \|GP_{N(F)^\perp}\varepsilon_n\|_A^2 = \sum_{n \in \mathcal{I}_2} \|G\varepsilon_n\|_A^2 \leq \sum_{n \in \mathcal{I}_2} \|G\varepsilon_n\|_A^2 + \sum_{n \in \mathcal{I}_1} \|G\varepsilon_n\|_A^2 = Tr(G^*AG),$$

since $\sum_{n \in \mathcal{I}_1} \|G\varepsilon_n\|_A \geq 0$.

Conversely, if $FG^*P_{\mathcal{S}} = P_{\mathcal{S}}$ then, by Lemma 5.3, $G = T^\dagger F + W + R$, where $W \in L(N(F)^\perp, \mathcal{S}^\perp)$ and $R \in L(N(F), \mathcal{H})$. Suppose that $G_0 = T^\dagger F + W_0 + R_0$, with $W_0 \in L(N(F)^\perp, \mathcal{S}^\perp)$ and $R_0 \in L(N(F), \mathcal{H})$, is an A -curve fitting process. Then, for every $W \in L(N(F)^\perp, \mathcal{S}^\perp)$ and $R \in L(N(F), \mathcal{H})$,

$$\sum_n \|G_0\varepsilon_n\|_A^2 \leq \sum_n \|(T^\dagger F + W + R)\varepsilon_n\|_A^2.$$

Let $R = 0$, since $\sum_n \|G_0 \varepsilon_n\|_A^2 = \sum_{n \in \mathcal{I}_2} \|(T^\dagger F + W_0) \varepsilon_n\|_A^2 + \sum_{n \in \mathcal{I}_1} \|R_0 \varepsilon_n\|_A^2$, it follows that for every $W \in L(N(F)^\perp, \mathcal{S}^\perp)$

$$(5.4) \quad \sum_{n \in \mathcal{I}_2} \|T^\dagger F \varepsilon_n + W_0 \varepsilon_n\|_A^2 \leq \sum_{n \in \mathcal{I}_2} \|T^\dagger F \varepsilon_n + W \varepsilon_n\|_A^2$$

Given $y \in \mathcal{S}$ ($y \neq 0$), let $v = F^* y$, then v satisfies $T^\dagger F v = y$. Given $n_0 \in \mathcal{I}_2$, let $\varepsilon_{n_0} = v/\|v\|$ and $\{\varepsilon_n\}_{n \neq n_0}$ such that $\{\varepsilon_n\}_{n \in \mathcal{I}_2}$ is an orthonormal basis of $N(F)^\perp$. For $x \in \mathcal{S}^\perp$, let $W_x \in L(N(F)^\perp, \mathcal{S}^\perp)$ such that, $W_x \varepsilon_{n_0} = -x$ and $W_x \varepsilon_n = W_0 \varepsilon_n$, for every $n \in \mathcal{I}_2, n \neq n_0$. By equation 5.4, it follows that

$$\left\| \frac{y}{\|v\|} - (-W_0 \varepsilon_{n_0}) \right\|_A \leq \left\| \frac{y}{\|v\|} - x \right\|_A,$$

for any $x \in \mathcal{S}^\perp$. Thus, $x_0 = -\|v\| W_0 \varepsilon_{n_0}$ is an A -WLSA of y . Then, by Corollary 4.3, the pair (A, \mathcal{S}^\perp) is compatible. \square

The following result gives an expression for a subset of A -curve fitting processes, notice that this expression does not depend on the frame \mathcal{V} , but only on the subspace \mathcal{S} .

Corollary 5.6. *Let $A \in L(\mathcal{H})^+$ be a trace class operator and \mathcal{S} be a closed subspace of \mathcal{H} such that the pair (A, \mathcal{S}^\perp) is compatible. Then $W \in \mathcal{P}_*(A, \mathcal{S})$ is an A -curve fitting process on \mathcal{S} .*

Proof. Suppose that $\mathcal{V} = \{v_n\}_{n \in \mathcal{I}}$ is a frame for \mathcal{S} , $(F, \mathcal{B} = \{e_n\}_{n \in \mathcal{I}}, \mathcal{K})$ is the synthesis operator of \mathcal{V} and $T = FF^*$. Let $Q \in \mathcal{P}(A, \mathcal{S}^\perp)$, by Theorem 5.5, if $G_0 = (I - Q)T^\dagger F$ then FG_0^* is an A -curve fitting process on \mathcal{S} . Since, $FG_0^* = F((I - Q)T^\dagger F)^* = TT^\dagger(I - Q^*) = (I - Q^*)$, it follows that if $W \in \mathcal{P}_*(A, \mathcal{S})$, then W is an A -curve fitting process on \mathcal{S} . \square

The previous corollary relates the set of A -curve fitting processes with the set of A -approximation processes. In fact, an A -curve fitting process exists if and only if an A -approximation process does. A similar result can be found in [17, Corollary of Theorem 5], for finite dimensional spaces.

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