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DIFFERENTIABILITY OF THE SOLUTIONS OF A SEMILINEAR ABSTRACT CAUCHY PROBLEM WITH RESPECT TO PARAMETERS

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Abstract. The Fréchet differentiability with respect to a parameter q of the solutions $z(t;q)$ of Cauchy problems of the form $\frac{d}{dt}z(t) = A(q)z(t) + F(q,t,z(t))$ is analyzed. Sufficient conditions on the operator $A(q)$ and on F are derived and the corresponding sensitivity equations for the Fréchet derivative $D_q z(t;q)$ are found.

1. Introduction

We consider the problem of continuous dependence an differentiability with respect to a parameter q of the solutions $z(t;q)$ of the semilinear abstract Cauchy problem

$$(\mathcal{P})_q \begin{cases} \frac{d}{dt}z(t) = A(q)z(t) + F(q,t,z(t)) & z(t) \in Z, \\ z(0) = z_0 & t \in [0,T] \end{cases}$$

where Z is a Banach space, $q \in Q_{ad} \subset Q$, a normed linear space (Q_{ad} is an open subset of Q), and $A(q)$ is the infinitesimal generator of an analytic semigroup $T(t;q)$ on Z for all $q \in Q_{ad}$. Z and Q are the state space and the parameter space, respectively, while Q_{ad} is called the admissible parameter set.

Identification problems associated to system $(\mathcal{P})_q$ and other similar type of equations ([2], [5], [7]) are usually solved by direct methods such as quasilinearization. These methods require that solutions be differentiable with respect to the parameter q. In addition, their numerical implementation require an approximation to the corresponding Fréchet derivative.

Problems of the type $\frac{d}{dt}z(t) = A(q)z(t) + u(t)$, where $A(q)$ generates a strongly continuous semigroup and $A(q) = A + B(q)$ where $B(q)$ is assumed to be bounded where studied by Clark and Gibson ([4]), Brewer ([1]). Burns et al ([3]) studied problems of the type $\frac{d}{dt}z(t) = Az(t) + F(q,t,z(t))$. The parameter q here did not appear in the linear part of the equation.

Here, we prove that, under certain conditions, the solutions of the general abstract Cauchy problem $(\mathcal{P})_q$ are Fréchet differentiable with respect to q and we find the corresponding sensitivity equations.

Key words and phrases. Abstract Cauchy problem, analytic semigroup, infinitesimal generator, Fréchet differentiability, Fréchet derivative.

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2. Preliminary Results

The following standing hypotheses are considered:

H1: There exist $\varepsilon_0 > 0$ such that the type of $T(t;q)$, call it w_q, is less than or equal to $-\varepsilon_0$ for all $q \in Q_{ad}$ and there exists $C_q > 0$ such that $\|T(t;q)\| \leq C_q e^{-\varepsilon_0 t}$ for all $t \geq 0$ and $q \in Q_{ad}$. The constant C_q depends on q but it can be chosen independent of q on compact subsets of Q_{ad}.

H2: $\mathcal{D}(A(q)) = D$ is independent of q and D is a dense subspace of Z.

We shall denote by Z_δ the space $D\left((-A(q))^\delta\right)$ imbedded with the norm of the graph of $-A(q))^\delta$. Since $0 \in \rho(A(q))$ it follows that this norm is equivalent to $\|z\|_{q,\delta} \doteq \|(-A(q))^\delta z\|$. Also, there exists a constant M_q such that $\|(-A(q))^\delta T(t;q)\| \leq M_q \frac{e^{-\epsilon_0 t}}{t^\delta}$, for all $t > 0$ (see [13], Theorem 2.6.13).

H3: There exists $\delta \in (0,1)$ such that

$$\|F(q,t_1,z_1) - F(q,t_2,z_2)\|_Z \leq L(|t_1 - t_2| + \|z_1 - z_2\|_{q,\delta})$$

for $(t_i, z_i) \in U$, where L can be chosen independent of q on any compact subset of Q_{ad}.

This last regularity condition guarantees existence and uniqueness of solutions of problem $(\mathcal{P})_q$, provided that the initial condition z_0 is in Z_δ. See [12] and [11] for details.

The next results can be easily proved by using the Closed Graph Theorem.

Lemma 1: *Under hypotheses H1 and H2, for any $q_1, q_2 \in Q_{ad}$ and $\delta \in (0,1)$ we have:*
i) $A(q_1)(-A(q_2))^{-\delta}$ *is bounded on* $Z_{1-\delta}$.
ii) $A(q_1)T(\cdot;q_2) \in L^1(0,\infty : \mathcal{L}(Z))$ and $A(q_1)T(\cdot;q_2) \in L^\infty(\eta,\infty;\mathcal{L}(Z))$, for each $\eta > 0$.
iii) $T(\cdot;q_2) \in L^1(0,\infty : \mathcal{L}(Z, Z_{q_1,\delta}))$ and $T(\cdot;q_2) \in L^\infty(\eta,\infty : \mathcal{L}(Z;Z_{q_1,\delta}))$, for each $\eta > 0$.

Note: This result implies that the operator $A(q_1)T(t;q_2)$ is bounded for each $t > 0$. However, no uniform bound can be found for t near zero. For $q_1 = q_2 = q$, it implies, in particular, that the derivative $\frac{d}{dt}T(t;q)$ of the solution operator of the homogeneous equation associated with $(\mathcal{P})_q$ is integrable near $t = 0$.

We will also assume that $A(q)$ satisfies the following hypothesis:

H4: For δ as in H3 and for any $q_1, q_2 \in Q_{ad}$ there are constants $M(q_1,q_2)$ and $C(q_1,q_2)$ both depending on q_1 and q_2, such that $\|(-A(q_1))^\delta(-A(q_2))^{-\delta}\|_{\mathcal{L}(Z)} \leq M(q_1,q_2)$,
$\|A(q_1)[A(q_2)]^{-1} - I\| \leq C(q_1,q_2)$ and $C(q_1,q_2) \to 0$ as $q_1 \to q_2$.

Note: It is sufficient to request that H4 be true for $\delta = 1$.
We also consider the hypothesis:

H4': For each $q_0 \in Q_{ad}$ there exists $C = C(q_0)$ such that

$$\|(A(q) - A(q_0))z\| \leq C\|q - q_0\|\,\|A(q_0)z\| \quad z \in D, \quad q \in Q_{ad}.$$

Theorem 2: *Assume H1-H4 hold. Then for any $q_0 \in Q_{ad}$* and $\varepsilon > 0$, there exists $\widetilde{\delta} > 0$ *such that*

$$\|A(q)T(\cdot,q_0)z - A(q_0)T(\cdot,q_0)z\|_{L^1(0,\infty;Z)} \leq \varepsilon\|z\|$$

for all $z \in Z$, *and for all* $q \in Q_{ad}$ *satisfying* $\|q - q_0\| < \widetilde{\delta}$, *that is*

$$\|A(q)T(\cdot, q_0) - A(q_0)T(\cdot, q_0)\|_{L^1(0,\infty;\mathcal{L}(Z))} \leq \varepsilon,$$

or equivalently, for every fixed $q_0 \in Q_{ad}$ *the mapping from* Q into $L^1(0,\infty;\mathcal{L}(Z))$ defined by

$$q \to A(q)T(\cdot, q_0)$$

is continuous on Q_{ad} .

The proof follows immediately using Lemma 1.

3. Main results

Recall that for $z_0 \in Z_\delta$, $z(t;q)$ satisfies

$$z(t;q) = T(t;q)z_0 + \int_0^t T(t-s;q)F(q,s,z(s;q))ds \doteq T(t;q)z_0 + S(t;q), \quad t \in [0,T].$$

Consider now the following standing hypothesis concerning the q-regularity of $\frac{d}{dt}T(t;q)$.

H5: The mapping $q \to A(q)T(\cdot;q_0)$ from Q into $L^1(0,\infty;\mathcal{L}(Z))$ is Fréchet differentiable at q_0 for all $q_0 \in Q_{ad}$ (under H1-H4, we already know that this mapping is continuous, by virtue of Theorem 2).

Theorem 3: *Suppose H1-H5 hold. It follows that*

i) The mapping $q \to T(\cdot;q)$ *from* $Q \to L^\infty(0,\infty;\mathcal{L}(Z))$ *is Fréchet differentiable at* q_0, *for each* $q_0 \in Q_{ad}$. *Moreover, for any* $t > 0$ *and* $h \in Q_{ad}$ *the q-Fréchet derivative of* $T(t;q)$ *evaluated at* $q_0 \in Q_{ad}$ *and applied to* $h \in Q$, *i.e.* $[D_qT(t;q_0)]h$, *is the solution* $v_h(t)$ *of the following linear IVP, the so called "sensitivity equation" for* $T(t;q)$, *in* $\mathcal{L}(Z)$

$$(S_1): \begin{cases} \frac{d}{dt}v_h(t) = A(q_0)v_h(t) + \left[D_qA(q)T(t;q_0)|_{q=q_0}\right]h \\ v_h(0) = 0, \end{cases}$$

and ii) for every $q_0 \in Q_{ad}$, $D_qT(\cdot;q_0) = D_qT(\cdot;q)|_{q=q_0} \in L^\infty(0,\infty;\mathcal{L}(Q;\mathcal{L}(Z)))$.

Proof: For $q_0 \in Q_{ad}$ we have

$$[D_qT(t;q_0)z_0](\cdot) = \int_0^t T(t-s;q_0)\left[D_qA(q)T(s;q_0)z_0|_{q=q_0}\right](\cdot)\,ds. \tag{2}$$

It remains to show the Fréchet differentiability of the mapping $q \to T(\cdot;q)$ when viewed as a mapping form Q into $L^\infty(0,\infty;\mathcal{L}(Z))$, i.e. in the stronger $L^\infty(0,\infty;\mathcal{L}(Z))$ norm. Let $\varepsilon > 0$, $t > 0$ and $q_0 \in Q_{ad}$. First note that for any $h \in Q$ with $\|h\| < \tilde{\delta}$, ($\tilde{\delta}$ as in Theorem 2) we have

$$\begin{aligned}\frac{d}{dt}[T(t;q_0+h)z_0 - T(t;q_0)z_0] &= A(q_0+h)T(t;q_0+h)z_0 - A(q_0)T(t;q_0)z_0 \\ &= A(q_0+h)[T(t;q_0+h)z_0 - T(t;q_0)z_0] + (A(q_0+h) - A(q_0))T(t;q_0)z_0.\end{aligned}$$

From Theorem 2 and [13] (Corollary 2.2) it follows that

$$T(t;q_0+h)z_0 - T(t;q_0)z_0 = \int_0^t T(t-s;q_0+h)\left(A(q_0+h) - A(q_0)\right)T(s;q_0)z_0\,ds. \tag{3}$$

and therefore for all $h \in Q$ with $\|h\| < \tilde{\delta}$, we have

$$\begin{aligned}\|T(t;q_0+h)z_0 - T(t;q_0)z_0\|_Z &\le \int_0^t M_{q_0+h} e^{-\varepsilon_0(t-s)} \|(A(q_0+h)T(s;q_0) - A(q_0)T(s;q_0))z_0\|_Z \, ds \\ &\le C \|(A(q_0+h)T(\cdot;q_0) - A(q_0)T(\cdot;q_0))z_0\|_{L^1(0,\infty;Z)} \\ &\le C\varepsilon \|z_0\|_Z,\end{aligned}$$

Thus for $t > 0$

$$\|T(t;q_0+h) - T(t;q_0)\|_{\mathcal{L}(Z)} \le C\varepsilon, \quad \text{for } \|h\| < \tilde{\delta}, \tag{4}$$

and, since the constant C above does not depend on t,

$$\|T(\cdot;q_0+h) - T(\cdot;q_0)\|_{L^\infty(0,\infty;\mathcal{L}(Z))} \le C\varepsilon, \quad \text{for } \|h\| < \tilde{\delta}.$$

The following estimate then follows

$$\begin{aligned}&\left\| T(t;q_0+h) - T(t;q_0) - \int_0^t T(t-s;q_0) \left[D_q A(q) T(s;q_0)|_{q=q_0} \right] h \, ds \right\|_{\mathcal{L}(Z)} \\ &\le (\varepsilon+1) C \left\| A(q_0+h)T(\cdot;q_0) - A(q_0)T(\cdot;q_0) - \left[D_q A(q) T(\cdot;q_0)|_{q=q_0} \right] h \right\|_{L^1(0,\infty;\mathcal{L}(Z))} \\ &\quad + \varepsilon C \left\| \left[D_q A(q) T(\cdot;q_0)|_{q=q_0} \right] h \right\|_{L^1(0,\infty;\mathcal{L}(Z))}.\end{aligned} \tag{5}$$

Now by (H5) for the given $\varepsilon > 0$ there exists $\xi > 0$ such that

$$\left\| A(q_0+h)T(\cdot;q_0) - A(q_0)T(\cdot;q_0) - \left[D_q A(q) T(\cdot;q_0)|_{q=q_0} \right] h \right\|_{L^1(0,\infty;\mathcal{L}(Z))} \le \varepsilon \|h\| \tag{6}$$

for $\|h\| < \xi$, and since $D_q A(q) T(\cdot;q_0)|_{q=q_0} \in \mathcal{L}\left(Q, L^1(0,\infty;\mathcal{L}(Z))\right)$ there exists M, $0 < M < \infty$ such that

$$\|D_q A(q) T(\cdot,q_0)|_{q=q_0}\|_{\mathcal{L}(Q,L^1(0,\infty,\mathcal{L}(Z)))} \le M. \tag{7}$$

Now, employing (6) and (7) in (5) we get that for $\|h\| < \min(\tilde{\delta}, \xi)$

$$\begin{aligned}&\left\| T(t;q_0+h) - T(t;q_0) - \int_0^t T(t-s;q_0) \left[D_q A(q) T(s;q_0)|_{q=q_0} \right] h \, ds \right\|_{\mathcal{L}(Z)} \\ &\le (\varepsilon+1) C\varepsilon \|h\| + \varepsilon C M \|h\| \le K\varepsilon \|h\|.\end{aligned}$$

Hence the mapping from Q into $L^\infty(0,\infty;\mathcal{L}(Z))$ defined by

$$q \to T(\cdot;q)$$

is Fréchet q-differentiable at q_0 and

$$\left[D_q T(t;q_0) \right](\cdot) = \int_0^t T(t-s;q_0) \left[D_q A(q) T(s;q_0)|_{q=q_0} \right](\cdot) \, ds. \tag{8}$$

Since $q_0 \in Q_{ad}$ is arbitrary, part (i) of the Theorem follows. To prove (ii) we first note that by H5, for $q_0 \in Q_{ad}$, there exists $C = C(q_0)$ such that for $h \in Q$

$$\|D_q A(q) T(\cdot;q_0)|_{q=q_0} h\|_{L^1(0,\infty;\mathcal{L}(Z))} \le C(q_0) \|h\|. \tag{9}$$

Now, it follows from (8) that for $t > 0$, $q_0 \in Q_{ad}$ and $h \in Q$, one has $\| [D_qT(t; q_0)]\ h\|_{\mathcal{L}(Z)} \leq \tilde{C}(q_0)\, \|h\|$. Thus $\|D_qT(t; q_0)\|_{\mathcal{L}(Q;\mathcal{L}(Z))} \leq \tilde{C}(q_0)$, and since $\tilde{C}(q_0)$ does not depend on $t > 0$, it follows that $D_qT(\cdot; q_0) \in L^\infty(0, \infty; \mathcal{L}(Q; \mathcal{L}(Z)))$.

■

Under slightly stronger assumptions on the mapping $q \to A(q)T(\cdot; q_0)$, it is possible to obtain the Lipschitz continuity of the mapping $q \to D_qT(\cdot; q_0)$ as a mapping from Q into $L^\infty(0, \infty; \mathcal{L}(Q; \mathcal{L}(Z)))$ and from Q into $L^\infty(0, \infty; \mathcal{L}(Q; \mathcal{L}(Z, Z_\delta)))$. In fact, consider the following hypothesis.

H6: The mapping $q \to D_qA(q)T(\cdot; q_0)$ from Q into $L^1(0, \infty; \mathcal{L}(Q; \mathcal{L}(Z)))$ is locally Lipschitz continuous at q_0, for all $q_0 \in Q_{ad}$.

THEOREM 4: *Let $q_0 \in Q_{ad}$ and assume hypotheses H1-H6 hold. Then the mapping $q \to D_qT(\cdot; q_0)$ from Q into $L^\infty(0, \infty; \mathcal{L}(Q; \mathcal{L}(Z)))$ is locally Lipschitz continuous at q_0.*

PROOF: Choose $h \in Q$ such that $\|h\| < \tilde{\delta}$ ($\tilde{\delta}$ as in Theorem 2) and denote $G_q(t; q_0)(\cdot) = D_qA(q)T(t; q_0)|_{q=q_0}(\cdot) \in \mathcal{L}(Q, \mathcal{L}(Z))$. Theorem 3 together with the appropriate choice of $\alpha(h)$, $0 \leq |\alpha(h)| \leq 1$, yield

$$\begin{aligned}
&\|D_qT(t; q_0 + h)(\cdot) - D_qT(t; q_0)(\cdot)\|_{\mathcal{L}(Q;\mathcal{L}(Z))} \\
&\leq M_{q_0+h}\, \|G_q(\cdot; q_0 + h) - G_q(\cdot; q_0)\|_{L^1(0,\infty;\mathcal{L}(Q;\mathcal{L}(Z)))} \\
&\quad + \|D_qT(\cdot; q_0 + \alpha(h)h)\|_{L^\infty(0,\infty;\mathcal{L}(Q;\mathcal{L}(Z)))}\, \|G_q(\cdot; q_0)\|_{L^1(0,\infty;\mathcal{L}(Q;\mathcal{L}(Z)))}\, \|h\| \\
&\leq C\|h\|.
\end{aligned}$$

The last inequality follows from H6 and Theorem 3(ii), and by the fact that $G_q(\cdot, q_0) \in L^1(0, \infty; \mathcal{L}(Q; \mathcal{L}(Z)))$, which is a result of H6. ■

In order to obtain the q-Frechét differentiability of $S(\cdot; q)$, we will need the local Lipschitz continuity of the mapping $q \to D_qT(\cdot; q_0)$ when viewed as a mapping from Q into $L^\infty(0, \infty; \mathcal{L}(Q; \mathcal{L}(Z; Z_\delta)))$. This can be achieved by requiring the following hypothesis.

H7: For every $q_0 \in Q_{ad}$, $D_qA(q)T(\cdot; q_0)|_{q=q_0} \in L^1(0, \infty; \mathcal{L}(Q; \mathcal{L}(Z; Z_\delta)))$ and the mapping $q \to D_qA(q)T(\cdot; q_0)$ from Q into $L^1(0, \infty; \mathcal{L}(Q; \mathcal{L}(Z; Z_\delta)))$ is locally Lipschitz continuous at q_0, for all $q_0 \in Q_{ad}$.

Clearly H7 implies H6 (since the Z_δ-norm is stronger that the Z-norm).

THEOREM 5: *Assume H1-H5 and H7 hold. Then, for all $q_0 \in Q_{ad}$, $D_qT(\cdot; q_0) \in L^\infty(0, \infty; \mathcal{L}(Q; \mathcal{L}(Z; Z_\delta)))$ and, the the mapping $q \to D_qT(\cdot; q)$ from Q into the space $L^\infty(0, \infty; \mathcal{L}(Q; \mathcal{L}(Z; Z_\delta)))$ is locally Lipschitz continuous at q_0.*

PROOF: For $t > 0$, $z \in Z$, $h \in Q$, it follows that

$$\begin{aligned}
\| [D_qT(t; q_0)h]\ z\|_{Z_\delta} &= \left\|(-A(q_0))^\delta\, ([D_qT(t; q_0)]\ h)\, z\right\|_Z \\
&= M_{q_0}\|h\|\, \|z\|_Z\, \|D_qA(q)T(\cdot; q_0)|_{q=q_0}\|_{L^1(0,t;\mathcal{L}(Q;\mathcal{L}(Z;Z_\delta)))} \\
&\leq C(q_0)\, \|h\|\, \|z\|_Z \qquad \text{(by virtue of H7)}
\end{aligned}$$

Hence, $\|D_qT(t; q_0)\, h\|_{\mathcal{L}(Z;Z_\delta)} \leq C(q_0)\, \|h\|$, and $\|D_qT(t; q_0)\|_{\mathcal{L}(Q;\mathcal{L}(Z;Z_\delta))} \leq C(q_0)$.

Since $C(q_0)$ does not depend on $t > 0$, it follows that $D_qT(\cdot; q_0) \in L^\infty(0,\infty; \mathcal{L}(Q; \mathcal{L}(Z; Z_\delta)))$. The Lipschitz continuity of this mapping is obtained following the same steps as in Theorem 4. ■

This result implies that $q \to T(\cdot; q)$ is Fréchet differentiable as a mapping from Q into $L^\infty(0,\infty; \mathcal{L}(Q; \mathcal{L}(Z; Z_\delta)))$. In fact, THEOREM 6: *Under the same hypotheses of Theorem*

5, $T(\cdot; q)$ is Fréchet differentiable at q_0, for each $q_0 \in Q_{ad}$, when viewed as a mapping from Q into $L^\infty(0,\infty; \mathcal{L}(Z; Z_\delta))$.

PROOF: For $h \in Q$ with $\|h\| < \tilde{\delta}$ so that $q_0 + \alpha h \in Q_{ad}$, α satisfying $|\alpha| \leq 1$, $\beta(h)$ appropriately chosen, $0 \leq |\beta(h)| \leq 1$, and any $t > 0$ it follows that

$$\begin{aligned}
&\|T(t; q_0 + h) - T(t; q_0) - [D_qT(t; q_0)]\, h\|_{\mathcal{L}(Z;Z_\delta)} \\
&\leq C(q_0)\, \|\beta(h)h)\|\, \|h\| \\
&\leq C(q_0)\, \epsilon\, \|h\|, \qquad \text{for } \|h\| < \epsilon, \text{ for all } \epsilon \text{ such that } 0 < \epsilon \leq \tilde{\delta}.
\end{aligned}$$

■

Note that Theorems 3 and 6 imply that the solution $z_h(t; q)$ of the linear homogeneous problem associated to $(\mathcal{P})_q$ is Fréchet differentiable with respect to q, both as a mapping into Z and into Z_δ, respectively. Theorems 4 and 5 imply, moreover, that the corresponding Fréchet derivatives are locally Lipschitz continuous.

The following generalization of Growall's Lemma for singular kernels will be needed later. Its proof can be found in [6], Lemma 7.1.1.

LEMMA 7: *Let L, T, δ be positive constants, $\delta < 1$, $a(t)$ a real valued, nonnegative, locally integrable function on $[0, T]$ and $\mu(t)$ a real-valued function on $[0, T]$ satisfying*

$$\mu(t) \leq a(t) + L \int_0^t \frac{\mu(s)}{(t-s)^\delta}\, ds, \quad t \in [0, T].$$

Then, there exists a constant K depending only on δ such that

$$\mu(t) \leq a(t) + KL \int_0^t \frac{a(s)}{(t-s)^\delta}\, ds, \quad t \in [0, T].$$

Before proving the Fréchet differentiability of the mapping $q \to S(\cdot; q)$ from $Q \to L^\infty(0, T; Z_\delta)$, we will show that if $F(q, t, z)$ satisfies appropriate regularity properties, such a mapping is locally Lipschitz continuous at q_0, for all $q_0 \in Q_{ad}$. This result will be needed later.

Consider the hypothesis:

- **H8:** The mapping $q \to F(q, \cdot; z)$ from Q into $L^\infty(0, T; Z)$ is locally Lipschitz continuous for all $z \in Z_\delta$ with Lipschitz constant independent of z on Z_δ-bounded sets.

THEOREM 8: *Let $q_0 \in Q_{ad}, z_0 \in D_\delta$ and assume H1-H5, H7 and H8 hold. Then the mapping $q \to S(\cdot; q)$ from $Q \to L^\infty(0, T; Z_\delta)$ is locally Lipschitz continuous at q_0.*

PROOF: Unsing Theorem 3 we write

$$\begin{aligned}
&S(t;q_0+h)-S(t;q_0)=\\
&\quad=\int_0^t T(t-s;q_0+h)\left[F(q_0+h,s,z(s;q_0+h))-F(q_0,s,z(s;q_0+h))\right]ds\\
&\quad+\int_0^t T(t-s;q_0+h)\left[F(q_0,s,z(s;q_0+h))-F(q_0,s,z(s;q_0))\right]ds\\
&\quad+\int_0^t D_qT(t-s;q_0+\beta(h)h)\,hF(q_0,s,z(s;q_0))\,ds,
\end{aligned}$$

where $q_0+h\in Q_{ad}$ for all $\|h\|\le\gamma_1$ and $\beta(h)$ is an appropriately selected constant satisfying $0\le|\beta(h)|\le 1$.
Using H8, H3 and Theorem 5 it then follows that

$$\begin{aligned}
&\|S(t;q_0+h)-S(t;q_0)\|_\delta\\
&\le\int_0^t \frac{M_{q_0+h}e^{-\varepsilon_0(t-s)}}{(t-s)^\delta}C_1\|h\|\,ds+\int_0^t \frac{M_{q_0+h}e^{-\varepsilon_0(t-s)}}{(t-s)^\delta}L\,\|z(s;q_0+h)-z(s;q_0)\|_\delta+C_2\|h\|\\
&\le C_3\|h\|\;+\;C_4\int_0^t\frac{\|\left[D_qT\left(s;q_0+\beta(h)h\right)h\right]z_0+S(s;q_0+h)-S(s;q_0)\|_\delta}{(t-s)^\delta}\,ds\\
&\le C_5\|h\|\;+\;C_4\int_0^t\frac{\|S(s;q_0+h)-S(s;q_0)\|_\delta}{(t-s)^\delta}\,ds.
\end{aligned}$$

Hence, by Lemma 7, there exist a constant K such that

$$\|S(t;q_0+h)-S(t;q_0)\|_\delta\le C_5\|h\|+KC_4C_5\|h\|\int_0^T\frac{1}{(t-s)^\delta}\,ds\doteq C_6\|h\|,\quad t\in[0,T],$$

provided $\|h\|\le\gamma_1$. The Theorem follows. ■

It is appropriate to note at this point that this result together with Theorem 6 imply that the mapping $q\to z(\cdot;q)$ from Q into $L^\infty(0,T;Z_\delta)$ is locally Lipschitz continuous at q_0. We proceed now to prove the Fréchet differentiability of the mapping $q\to S(t;q)$, corresponding to the nonlinear part of problem $(\mathcal{P})_q$. For that purpose, we consider the following hypothesis.

H9: The mapping $(q,z(\cdot))\to F(q,\cdot,z(\cdot))$ from $Q_{ad}\times L^1(0,T;Z_\delta)$ into $L^\infty(0,T;Z)$ is Fréchet differentiable in both variables, the mapping $(q,z(\cdot))\to F_q(q,\cdot,z(\cdot))$ from $Q\times L^\infty(0,T;Z_\delta)$ into $L^\infty(0,T:\mathcal{L}(Q;Z_\delta))$ is locally Lipschitz continuous with respect to q and z, with Lipschitz constant independent of z on Z_δ-bounded sets and $F_z(q,\cdot,z(\cdot;q))\in L^\infty(0,T;\mathcal{L}(Z;Z_\delta))$.

Clearly H9 is stronger than H8.
THEOREM 9: *Let $q_0\in Q_{ad}$, $z_0\in D_\delta$ and suppose H1-H5, H7 and H9 hold. Then the mapping $q\to S(t;q)=\int_0^t T(t-s;q)F(q,s,z(s;q))\,ds$ from $Q\to L^\infty(0,T;Z_\delta)$ is Fréchet differentiable at q_0. Moreover, for any $t\in[0,T]$, and any $h\in Q_{ad}$, $[D_qS(t;q_0)]h\doteq w_h(t)$ satisfies the integral equation*

$$w_h(t) = \int_0^t \Big\{ T(t-s;q_0)\Big[F_q(q_0,s,z(s;q_0))h + F_z\left(q_0,s,z(s;q_0)\right)\left[D_qT(s;q_0)z_0\right]h$$

(10)

$$+ F_z(q_0,s,z(s;q_0))w_h(s)\Big] + \left[D_qT(t-s;q_0)F(q_0,s,z(s;q_0))\right]\, h\Big\}\, ds,$$

and $w_h(t)$ is the solution of the following non-homogeneous linear IVP, the so-called "sensitivity equation" for $S(t;q)$, in Z:

$$(S_2)\begin{cases} \frac{d}{dt}w_h(t) = \big(A(q_0) + F_z(q_0,t,z(t;q_0))\big)w_h(t)F_q(q_0,t,z(t;q_0))h + \\ \qquad + F_z(q_0,t,z(t;q_0))[D_qT(t;q_0)z_0]h + \int_0^t D_qA(q)T(t-s;q_0)\big|_{q=q_0} h\, F\left(q_0,s,z(s;q_0)\right)\, ds \\ w_h(0) = 0. \end{cases}$$

Proof: That the solution $w_h(t)$ of(S_2) satisfies (10)follows immediately (S_1) in Theorem 2 and the fact that $[D_qT(0;q_0)z]h = 0$ for $z \in Z$ and $h \in Q$.

We write

$$\begin{aligned} &S(t;q_0+h) - S(t;q_0) - w_h(t) = \\ &= \int_0^t T(t-s;q_0)\big[F(q_0+h,s,z(s;q_0)) - F(q_0,s,z(s;q_0)) - F_q(q_0,s,z(s;q_0))h\big]\, ds \\ &\quad + \int_0^t T(t-s;q_0)\Big[F(q_0,s,z(s;q_0+h)) - F(q_0,s,z(s;q_0)) \\ &\qquad\qquad - F_z(q_0,s,z(s;q_0))\big(z(s;q_0+h) - z(s;q_0)\big)\Big]\, ds \\ &\quad + \int_0^t T(t-s;q_0)F_z(q_0,s,z(s;q_0))\big[S(s;q_0+h) - S(s;q_0) - w_h(s)\big]\, ds \\ &\quad + \int_0^t T(t-s;q_0)F_z(q_0,z(s;q_0))\Big[\left[D_qT(s;q_0+\alpha(h)h)z_0\right]h - \left[D_qT(s;q_0)z_0\right]h\Big]\, ds \\ &\quad + \int_0^t \Big\{T(t-s;q_0+h)F(q_0,s,z(s;q_0)) - T(t-s;q_0)F(q_0,s,z(s;q_0)) \\ &\qquad\qquad - \big[D_qT(t-s;q_0)F(q_0,s,z(s;q_0))\big]h\Big\}\, ds \\ &\quad + \int_0^t T(t-s;q_0+h)\big[F(q_0+h,s,z(s;q_0+h)) - F(q_0,s,z(s;q_0))\big]\, ds \\ &\quad - \int_0^t T(t-s;q_0)\big[F(q_0+h,s,z(s;q_0)) - 2F(q_0,s,z(s;q_0)) + F(q_0,s,z(s;q_0+h))\big]\, ds \\ &\doteq \sum_{i=1}^{7} I_i, \end{aligned}$$

where I_i is the i^{th} term in the expression written above. Here, $\alpha(h)$ is an appropriately chosen constant satisfying $0 \le |\alpha(h)| \le 1$.

In what follows, C_i will denote a generic finite positive constant depending on q_0. Let $\gamma_1 > 0$ be such that $q_0 + \eta \in Q_{ad}$ for all $\eta \in Q$ satisfying $\|\eta\| < \gamma_1$. Then for any $h \in Q_{ad}$

with $\|h\| < \gamma_1$ it follows, by virtue of Theorem 8 and hypothesis H9 that there exist positive constants C_1, C_2 and L, such that:

$$\begin{aligned} \| I_6 + I_7\|_\delta &\leq C_1\|h\|^2 + \int_0^t \frac{L}{(t-s)^\delta}\Big(|\alpha_1(h) - \alpha_3(h)|\,\|h\| + \|z(s;q_0+h) - z(s;q_0)\|_\delta\Big)\|h\|\,ds \\ &\quad + \int_0^t \frac{C_2}{(t-s)^\delta}\,\|z(s;q_0+h) - z(s;q_0)\|_\delta\,\|h\|\,ds \\ &\leq C_3 \parallel h \parallel^2, \quad \text{provided } \parallel h \parallel \leq \gamma_1, \end{aligned} \tag{11}$$

where the last inequality follows from the Lipschitz continuity of the mapping $q \to z(\cdot;q)$ from Q into $L^\infty(0,T;Z_\delta)$ at q_0. Now let ε be a fixed positive constant. It follows from H9 that there exist $\gamma_2 > 0$ and $\gamma_3 > 0$ such that

$$\|I_1\|_\delta \leq \int_0^t \frac{C_4}{(t-s)^\delta}\,\varepsilon\|h\|\,ds \leq C_5\,\varepsilon\,\|h\|, \tag{12}$$

provided $\|h\| \leq \gamma_2$, and also

$$\begin{aligned} \|I_2\|_\delta &\leq \int_0^t \frac{C_6\,\varepsilon}{(t-s)^\delta}\,\|z(s;q_0+h) - z(s;q_0)\|_Z\,ds \\ &\leq C_7\,\varepsilon\,\|h\|, \quad \text{provided} \|h\| \leq \gamma_3. \end{aligned} \tag{13}$$

Also, by using H9 we have that $F_z\,(q_0,\cdot,z(\cdot;q_0)) \in L^\infty(0,T;\mathcal{L}(Z;Z_\delta))$, and therefore there exists a constant C_8 such that

$$\|I_3\|_\delta \leq C_8 \int_0^t \frac{\big\|S(s;q_0+h) - S(s;q_0) - w_h(s)\big\|_\delta}{(t-s)^\delta}\,ds. \tag{14}$$

On the other hand, the local Lipschitz continuity of $D_qT(\cdot\,;q_0)$ (Theorem 4), implies the existence of two finite positive constants C_9 and γ_4 such that

$$\|I_4\|_\delta \leq \int_0^t \frac{C_9}{(t-s)^\delta}|\alpha(h)|\,\|h\|^2\,ds \leq C_{10}\|h\|^2, \text{ provided } \|h\| \leq \gamma_4. \tag{15}$$

Finally from Theorem 6 and H9, there are two finite positive constants C_{10} and γ_5 such that

$$\|I_5\|_\delta \leq C(q_0)\varepsilon\|h\| \int_0^t \|F(q_0,s,z(s;q_0))\|_Z\,ds \leq C_{10}\,\varepsilon\,\|h\|, \tag{16}$$

provided $\|h\| \leq \gamma_5$.

From (11)-(16) we conclude that there exist finite positive constants C_{11}, C_{12}, and γ such that for $t \in [0,T]$ and $h \in Q_{ad}$ with $\|h\| \leq \gamma$

$$\begin{aligned} \|S(t;q_0+h) - S(t;q_0) - w_h(t)\|_\delta &\leq C_{11}\,\varepsilon\|h\| + \\ &\quad + C_{12} \int_0^t \frac{\|S(s;q_0+h) - S(s;q_0) - w_h(s)\|_\delta}{(t-s)^\delta}\,ds. \end{aligned}$$

Hence, applying Lemma 7 we conclude that

$$\|S(t;q_0+h)-S(t;q_0)-w_h(t)\|_\delta \leq C_{11}\,\varepsilon\|h\| + KC_{12}C_{11}\varepsilon\|h\| \int_0^t \frac{1}{(t-s)^\delta}\,ds$$
$$\leq C_{13}\varepsilon\|h\|, \quad t\in[0,T], \quad \|h\|\leq\gamma.$$

hence the mapping $q \to S(\cdot;q)$ from $Q \to L^\infty(0,T;Z_\delta)$ is Fréchet differentiable at q_0 and $w_h(t)$ is the Fréchet derivative of $S(t;q)$ at q_0, i.e. $D_qS(t;q_0)=w_h(t)$. ■

Theorem 10: *Under the same hypotheses of Theorem 9, the mapping $q \to z(\cdot;q)$ from the admissible parameter set Q_{ad} into the solution space $L^\infty(0,T;Z_\delta)$, is Fréchet differentiable at q_0. Moreover, for any $h\in Q$, $t\in[0,T]$, the q-Fréchet derivative of $z(t;q)$ evaluated at q_0 and applied to h, i.e. $[D_qz(t;q_0)]h$ is the solution $v_h(t)$ of the following linear non-homogeneous initial value problem in Z, the sensitivity equation for $z(t;q)$*

$$(S)\begin{cases} \frac{d}{dt}v_h(t) = \big(A(q_0)+F_z(q_0,t,z(t;q_0))\big)\,v_h(t) \;+\; F_q(q_0,t,z(t;q_0))h + \\ \qquad + D_qA(q)T(t;q_0)z_0\big|_{q=q_0}h \;+\; \int_0^t D_qA(q)T(t-s;q_0)\big|_{q=q_0}h\,F\,(q_0,s,z(s;q_0))\;ds \\ \\ v_h(0)=0. \end{cases}$$

Proof: The Fréchet differentiability of $z(t;q) = T(t;q)z_0 + S(t;q)$ follows immediately from Theorems 6 and 9 and the sensitivity equation is readily obtained by combining the sensitivity equations (S_1) and (S_2). ■

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