

Fully discrete schemes for monotone optimal control problems

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Abstract

In this article we study a finite horizon optimal control problem with monotone controls. We consider the associated Hamilton-Jacobi-Bellman (HJB) equation which characterizes the value function.

We consider the totally discretized problem by using the finite element method to approximate the state space Ω . The obtained problem is equivalent to the resolution of a finite sequence of stopping-time problems.

The convergence orders of these approximations are proved, which are in general $(h + \frac{k}{\sqrt{h}})^\gamma$ where γ is the Hölder constant of the value function u . A special election of the relations between the parameters h and k allows to obtain a convergence of order $k^{\frac{2}{3}\gamma}$, which is valid without semiconcavity hypotheses over the problem's data.

We show also some numerical implementations in an example.

1 Introduction

We study a fully discrete scheme for the numerical resolution of the infinite horizon monotone optimal control problem, through the analysis of the associated finite horizon problem as in [13].

The considered system is governed by the following differential equation

$$\begin{cases} \dot{y}(s) = g(y(s), \alpha(s)), & s > 0, \\ y(0) = x, \end{cases} \quad (1.1)$$

where $\alpha(\cdot)$ is the control function, which throughout this article is restrained to be a non-decreasing monotone function defined in the set $[0, +\infty)$, with values in $[0, 1]$. For each $a \in [0, 1]$ let $\mathcal{A}(a)$ be the set of these functions with initial values higher or equal to a , i.e. $\alpha(0) \geq a$. At a point $s > 0$, the vector $y(s) \in \mathbb{R}^\nu$ is the state corresponding to the control $\alpha(\cdot)$ employed; $x \in \Omega$ is the initial state, where $\Omega \subset \mathbb{R}^\nu$ is an open set. We assume that the evolution of the system always stays in Ω , no matter which control is selected.

The performance of the employed control is measured through the functional J

$$J(x, \alpha(\cdot)) := \int_0^\infty f(y(s), \alpha(s)) e^{-\lambda s} ds,$$

where $f : \Omega \times [0, 1] \rightarrow \mathbb{R}$ is the instantaneous cost and $\lambda > 0$ is the discount factor. The problem consists in finding for each pair $(x, a) \in \Omega \times [0, 1]$, a control $\bar{\alpha} \in \mathcal{A}(a)$ that attains the minimum of the functional J . Therefore, the value function u is

$$u(x, a) := \inf_{\alpha \in \mathcal{A}(a)} J(x, \alpha(\cdot)), \quad (1.2)$$

which allows to construct optimal or suboptimal policies in feedback [1, 2].

We assume the following Lipschitz continuity and boundedness hypotheses on the functions g and f : there exist positive constants L_g, M_g, L_f and M_f such that $\forall x, \bar{x} \in \Omega, \forall a, \bar{a} \in [0, 1]$,

$$\|g(x, a) - g(\bar{x}, \bar{a})\| \leq L_g (\|x - \bar{x}\| + |a - \bar{a}|), \quad \|g(x, a)\| \leq M_g, \quad (1.3)$$

$$|f(x, a) - f(\bar{x}, \bar{a})| \leq L_f (\|x - \bar{x}\| + |a - \bar{a}|), \quad |f(x, a)| \leq M_f. \quad (1.4)$$

Under these assumptions, using classic arguments it can be proven that the function u is bounded and Hölder continuous in both variables with Hölder constant γ :

$$\begin{cases} \gamma = 1 & \text{if } \lambda > L_g \\ \gamma = \frac{\lambda}{L_g} & \text{if } \lambda < L_g \\ \gamma \in (0, 1) & \text{if } \lambda = L_g \end{cases} \quad (1.5)$$

We consider the quasi-variational HJB inequality associated to the problem, given by the equation

$$\min \left(Lu(x, a), \frac{\partial u(x, a)}{\partial a} \right) = 0, \quad \text{in } \Omega \times (0, 1), \quad (1.6)$$

where

$$Lu(x, a) = \frac{\partial u(x, a)}{\partial x} g(x, a) + f(x, a) - \lambda u(x, a),$$

and the boundary condition

$$u(x, 1) = \int_0^\infty f(\eta(s), 1) e^{-\lambda s} ds, \quad (1.7)$$

where $\eta(s)$ is the trajectory corresponding to the control $\alpha \equiv 1$, and x is the initial value. In general, this equation does not admit a solution in $C^1(\Omega \times (0, 1))$, so the notion of viscosity solutions comes into play (see [8]). Specifically, u is the unique viscosity solution of the HJB equation (1.6) with boundary conditions (1.7) (see [1]).

1.1 Discretization in time and control space

We would like to obtain discretization schemes in order to numerically solve the equation (1.6). For this aim, we introduce an auxiliary optimal control problem where the policies have the additional restriction of being uniformly step functions with values in a discrete, equi-spaced set. Specifically, the control variable a takes values in the set

$$I_h := \left\{ ih \mid i = 0 \dots \frac{1}{h} \right\}.$$

We also define $I_h(a) = I_h \cap [a, 1]$. In the space $C(\Omega \times I_h)$ we consider the operators

$$\begin{aligned} (A^{h,b}(w))(x, a) &= (1 - \lambda h)w(x + hg(x, a), b) + hf(x, a), \\ (A^h(w))(x, a) &= \min_{b \in I_h(a)} (A^{h,b}(w))(x, a), \end{aligned} \quad (1.8)$$

arising from the discretization of the HJB equation.

Having considered a discretization in time, we introduce the consistent problem of finding in the functional space $C(\Omega \times I_h)$

$$\text{Problem } P^h : \text{ Find the fixed point } u^h \text{ of the operator } A^h \quad (1.9)$$

From (1.8) it turns out that A^h is a contractive operator if $0 < h < \frac{1}{\lambda}$. From this property it is straightforward to prove that (1.9) has a unique solution u^h , which is bounded and uniformly Hölder continuous in the first variable (see [4]). If (1.3) and (1.4) are satisfied, the same techniques give the Hölder continuity in both variables.

1.2 Fully discrete infinite horizon problem

The previously presented discretization allowed us to obtain important convergence results (see [13]). Nevertheless, in order to obtain numerical methods to estimate u^h it is also necessary the discretization in the state variables. We obtain a solution u_k^h totally discrete, considering a discretization in these variables by the finite element method.

Discretization elements

Let $\{S_j^k\}$ be a family of triangulations of Ω , i.e. a set of simplices that approximate Ω in the following sense:

$\Omega_k = \bigcup_j S_j^k$ is a polyhedron of \mathbb{R}^n such that the following properties hold:

$$\max_j (\text{diam } S_j^k) = k. \quad (1.10)$$

$$\exists h_0 > 0 \text{ such that } x + hg(x, a) \in \Omega_k, \forall x \in \Omega_k, \forall a \in I_h(0), \forall h < h_0. \quad (1.11)$$

$$\begin{aligned} \Omega_k \rightarrow \Omega \text{ when } k \rightarrow 0, \text{ in the following sense: } \Omega_k \subset \Omega \text{ and} \\ \forall K \text{ compact contained in } \Omega, \exists \bar{k}(K)/K \subset \Omega_k \quad \forall k \leq \bar{k}(K). \end{aligned} \quad (1.12)$$

If d_i is the diameter of the simplex S_i^k , then $\exists \chi_1 > 0$ such that for every simplex of Ω_k , there exists a sphere of radius

$$r \leq \chi_1 d_i. \quad (1.13)$$

in the interior of the simplex. Furthermore there exists M , independent of the discretization, such that

$$\frac{k}{d_i} \leq M \quad \forall i. \quad (1.14)$$

We consider the set W_k of functions $w : \Omega_k \times I_h \rightarrow \mathbb{R}$, $w(\cdot, a)$ continuous in Ω_k , with $\frac{\partial w(\cdot, a)}{\partial x}$ constant in the interior of each simplex of Ω_k (that is, w is a linear finite element).

Let V^k be the union over j of the vertices of the family of simplices $\{S_j^k\}$. If N is the cardinality of V^k we consider an appropriate ordering: $V^k = \{x^i, i = 1, \dots, N\}$.

Remark 1.1. It is obvious that any $w \in W_k$ is completely determined by the values $w(x^i, a)$, $i = 1, \dots, N$, $a \in I_h$.

Remark 1.2. In order to accomplish simplicity in the notation and clarity in the arguments, we will use the letters C, M, K to denote arbitrary constants (whose values depend on the context where they appear and on the data of the problem, the constants $\lambda, M_g, M_f, L_g, L_f$, etc.) but not on the parameters of discretization h, k or of regularization ρ , etc.

We define the operator A_k^h the following way

$$A_k^h : W_k \rightarrow W_k$$

$$(A_k^h w)(x^i, a) = \min_{b \in I_h(a)} \{ (1 - \lambda h)w(x^i + hg(x^i, a), b) + hf(x^i, a) \}, \quad \forall x^i \in V^k, \forall a \in I_h. \quad (1.15)$$

Remark 1.3. The operator A_k^h is merely the restriction of A^h to the space W_k .

The final problem to be resolved numerically is the following

$$\textit{Problem } P_k : \text{ Find the fixed point of the operator } A_k^h \quad (1.16)$$

It is clear that A_k^h is a contractive operator in W^k with the norm

$$\|w\| = \max_{i,j} |w(x^i, a^j)|$$

that is $\forall w, \bar{w} \in W_k$

$$\|A_k^h(w) - A_k^h(\bar{w})\| \leq (1 - \lambda h)\|w - \bar{w}\| \quad (1.17)$$

and therefore it has an unique fixed point, say u_k^h . Then

$$u_k^h = A_k^h(u_k^h).$$

Remark 1.4. The problem P_k is a nonlinear fixed point problem with a very special structure. It is equivalent to a stochastic control problem over a Markov chain (the reason for this relationship is the existence of a *discrete maximum principle* (see [6, 7, 9]), valid for the schemes of discretization used in the definitions of A_k^h). In particular, it can be easily seen that it represents a finite sequence of concatenated optimal stopping times. Therefore it can be solved by applying specially designed methods for those problems (Picard or Howard, see [9], [14]).

The main result of this paper is the $\left(h + \frac{k}{\sqrt{h}}\right)^\gamma$ - convergence of the fully discretized infinite horizon problem, where γ is the Hölder constant of the value function u , and is stated in Theorem 3.1.

The paper is organized as follows. In section 2 we introduce the associated finite horizon problem, its discretization in time and the fully discretized problem. In section 3 we state the main result of this paper, namely the convergence of the fully discrete scheme in the infinite horizon case and state and prove some necessary results. In section 4 we prove the main result and make some remarks about the choice of the discretization parameters. In section 5 we show some numerical implementations. Finally, in section 6 we state the conclusions.

2 The associated finite horizon problem

With the purpose of obtaining some technical results on convergence, we consider a similar problem whose main difference is that it deals with finite horizon. The problem consists in finding the value function u_T , defined in the following way: $\forall t \in [0, T], \forall x \in \Omega, \forall a \in [0, 1]$,

$$u_T(t, x, a) := \inf_{a \in \mathcal{A}_T(a)} J_T(t, x, \alpha(\cdot)), \quad (2.1)$$

where $\mathcal{A}_T(a)$ is the set of non-decreasing functions defined in $[t, T)$ with values in $[a, 1]$, and the functional J_T is defined by

$$J_T(t, x, \alpha(\cdot)) := \int_t^T f(y(s), \alpha(s)) e^{-\lambda(s-t)} ds \quad (2.2)$$

with $y(\cdot)$ being the solution of (1.1) with initial condition $y(t) = x$.

If conditions (1.3) and (1.4) hold, then

$$|u_T(t, x, a)| \leq \frac{M_f}{\lambda} (1 - e^{-\lambda t})$$

and

$$|u_T(t, x, a) - u_T(t, \bar{x}, \bar{a})| \leq L_T (\|x - \bar{x}\| + |a - \bar{a}|)$$

where $L_T = \frac{L_f}{\lambda - L_g}$ if $\lambda > L_g$, $L_T = \frac{L_f}{L_g - \lambda} e^{(L_g - \lambda)T}$ if $\lambda < L_g$, and $L_T = TL_f$ if $\lambda = L_g$. Also, there exists a positive constant M_T such that

$$|u_T(t, x, a) - u_T(\bar{t}, x, a)| \leq M_T |t - \bar{t}|.$$

The proof of these properties makes use of classic techniques.

For the finite horizon case, the associated HJB equation takes the form

$$\min \left(Lu_T(t, x, a), \frac{\partial u_T(t, x, a)}{\partial a} \right) = 0 \quad \text{in } (0, T) \times \Omega \times (0, 1), \quad (2.3)$$

where

$$Lu_T = \frac{\partial u_T}{\partial t} + \frac{\partial u_T}{\partial x} g + f - \lambda u_T,$$

with final condition

$$u_T(T, x, a) = 0, \quad \forall (x, a) \in \Omega \times [0, 1],$$

and boundary condition

$$u_T(t, x, 1) = \int_t^T f(\eta(s), 1) e^{-\lambda(s-t)} ds, \quad (2.4)$$

where $\eta(s)$ is the trajectory with initial value $\eta(t) = x$ corresponding to the control $\alpha \equiv 1$.

The viscosity solution of the equation (2.3) is defined similarly to the infinite horizon case (see [8]), proving that the value function u_T is the unique viscosity solution.

A different approach to the finite horizon problem with monotone controls has been made in [12] establishing a Pontryagin Maximum Principle.

In a similar way to the infinite horizon problem, we consider for the finite horizon case a discretization in time. Let $h > 0$ and $I_h, I_h(a)$ as in the infinite horizon case.

Remark 2.1. In what follows, we consider that h^{-1} is an integer and that the horizon $T = \mu h$, with μ an integer.

In order to find the solution of (2.3) we employ recursive approximation schemes. A natural discretization (by finite differences) of the HJB equation gives the scheme

$$\begin{cases} u_T^h(\mu, x, a) = 0, \quad \forall x \in \Omega, \forall a \in I_h, \\ u_T^h(n-1, x, a) = \min_{b \in I_h(a)} \{(1 - \lambda h)u_T^h(n, x + hg(x, a), b) + hf(x, a)\}, \end{cases} \quad (2.5)$$

for $n = 1, \dots, \mu$.

In [13] we proved the following

Lemma 2.1. [13, Lemma 4.1] *Under the hypotheses (1.3) and (1.4), we have*

$$|u(x, a) - u_T(t, x, a)| \leq \frac{M_f}{\lambda} e^{-\lambda(T-t)}. \quad (2.6)$$

and the

Theorem 2.1. [13, Theorem 3.5]

$$|u_T(nh, x, a) - u_T^h(n, x, a)| \leq C\phi(n)h, \quad (2.7)$$

where $\phi(n)$ is defined by

$$\phi(n) = \begin{cases} e^{(L_g - \lambda)T + \lambda nh} & \text{if } L_g > \lambda, \\ Te^{L_g nh} & \text{if } L_g = \lambda, \\ e^{L_g nh} & \text{if } L_g < \lambda \end{cases}$$

and $C > 0$ is a constant independent of h .

We define now $u_{k,T}^h$, the function of optimal cost totally discrete for the problem with finite horizon, where

$$u_{k,T}^h(n, \cdot, \cdot) \in W_k \quad \forall n = 0, \dots, \mu$$

is determined by the recursive scheme

$$\begin{cases} u_{k,T}^h(n-1, \cdot, \cdot) = (A_k^h u_{k,T}^h)(n, \cdot, \cdot) & n = 1, \dots, \mu \\ u_{k,T}^h(\mu, \cdot, \cdot) \equiv 0, \end{cases}$$

for $0 \leq n \leq \mu$. Here, the operator A_k^h is the one defined in (1.15).

3 Main Result. Convergence of the totally discrete problem

The central result of this section gives a general estimation of the error with respect to the parameters of discretization h and k and the data of the problem.

Theorem 3.1. *Let us suppose that (1.3), (1.4) and the general properties of the triangulation hold, detailed in section 1. Thus there exists a constant M independent of h, k such that*

$$|u(x, a) - u_k^h(x, a)| \leq \begin{cases} M \left(h + \frac{k}{\sqrt{h}} \right), & \text{if } L_g < \lambda \\ M \left(h + \frac{k}{\sqrt{h}} \right)^\gamma, & \text{with } \gamma = \frac{\lambda}{L_g} \quad \text{if } L_g > \lambda \\ M \left(h + \frac{k}{\sqrt{h}} \right)^\gamma, & \text{with } \gamma \in (0, 1) \quad \text{if } L_g = \lambda \end{cases} \quad (3.1)$$

The procedure used to obtain the estimation of $|u(x, a) - u_k^h(x, a)|$ is based in the decomposition in the following terms

$$\begin{aligned} |u(x, a) - u_k^h(x, a)| &\leq |u(x, a) - u_T(0, x, a)| + |u_T(0, x, a) - u_T^h(0, x, a)| \\ &\quad + |u_T^h(0, x, a) - u_{k,T}^h(0, x, a)| + |u_{k,T}^h(0, x, a) - u_k^h(x, a)|. \end{aligned} \quad (3.2)$$

3.1 Preliminary results

Note 3.1. For the proofs of the next two Lemmas, in order to simplify the exposition of the arguments, we will work only with the case $\Omega = \mathbb{R}^\nu$. The general case can be treated without fundamental changes using the perturbation techniques of the domain described in [10].

Lemma 3.1. *If (1.3), (1.4) and (1.11) are satisfied, the following holds*

$$|u_{k,T}^h(0, x, a) - u_k^h(x, a)| \leq \frac{M_f}{\lambda} e^{-\lambda T} \quad (3.3)$$

Proof. A_k^h is a contractive operator, whose fixed point is u_k^h , that is,

$$u_k^h = A_k^h u_k^h,$$

$u_{k,T}^h$ is also defined in terms of A_k^h by

$$u_{k,T}^h = (A_k^h)^\mu(w_T),$$

being $w_T = 0$. By virtue of (1.17), we have

$$|u_{k,T}^h(0, x, a) - u_k^h(x, a)| \leq (1 - \lambda h)^\mu |u_k^h|,$$

then

$$|u_{k,T}^h(0, x, a) - u_k^h(x, a)| \leq \frac{M_f}{\lambda} e^{-\lambda T}. \quad (3.4)$$

□

Lemma 3.2. *Under the hypotheses (1.3), (1.4), (1.11), (1.13) and (1.14), we have*

$$\max_{0 \leq n \leq \mu} |u_T^h(n, x, a) - u_{k,T}^h(n, x, a)| \leq M \phi(T) \frac{k}{\sqrt{h}},$$

with

$$\phi(T) = \begin{cases} 1 & \text{if } L_g < \lambda \\ e^{(L_g - \lambda)T} & \text{if } L_g > \lambda \\ T & \text{if } L_g = \lambda. \end{cases} \quad (3.5)$$

To prove this lemma we will need some auxiliary results, which will be obtained next. It will be necessary to decompose

$$|u_T^h(n, x, a) - u_{k,T}^h(n, x, a)|$$

in three different addends by intercalating specially defined regularized functions using a regular function and the convolution operator. In the following propositions we will seize the regularity properties of these functions.

Definition of $u_{T,\rho}^h$, regularized function of u_T^h

Definition. We consider $\beta(\cdot) \in C^\infty(\mathbb{R}^\nu)$ such that

$$\beta(x) \geq 0 \quad \forall x, \quad \text{support of } \beta \subset B_1 = \{x \in \mathbb{R}^\nu / \|x\| \leq 1\}, \quad \int_{\mathbb{R}^\nu} \beta(x) dx = 1.$$

We define $\forall \rho \in \mathbb{R}^+$

$$\beta_\rho(x) = \frac{1}{\rho^\nu} \beta\left(\frac{x}{\rho}\right) \geq 0.$$

We obtain a regular approximation of u_T^h by considering the convolution (with respect to the spatial variables) with the function $\beta_\rho(\cdot)$, that is $\forall n = 0, \dots, \mu, \forall a \in I_h, x \in \Omega$

$$u_{t,\rho}^h(n, x, a) = \int_{B(\rho)} u_T^h(n, x - y, a) \beta_\rho(y) dy = (u_T^h(n, \cdot, a) * \beta_\rho)(x). \quad (3.6)$$

Definition of $\tilde{u}_{T,\rho}^h$, linear interpolate of $u_{T,\rho}^h$

Definition. For each $n = 0, 1, \dots, \mu$, we define $\tilde{u}_{T,\rho}^h(n, \cdot, \cdot) \in W_k$ by assigning in the nodes of interpolation in a natural way the values

$$\tilde{u}_{T,\rho}^h(n, x^i, a) = u_{T,\rho}^h(n, x^i, a), \quad \forall n = 0, \dots, \mu, \forall i = 1, \dots, N, \forall a \in I_h. \quad (3.7)$$

Proposition i. For u_T^h the following estimate holds

$$|u_T^h(n, x, a) - u_{T,\rho}^h(n, x, a)| \leq L_u \rho.$$

Proof. by the definition of convolution we have from (3.6)

$$|u_T^h(n, x, a) - u_{T,\rho}^h(n, x, a)| \leq \int_{B(\rho)} |u_T^h(n, x, a) - u_T^h(n, x - y, a)| \beta_\rho(y) dy;$$

since u_T^h is Lipschitz continuous, we have

$$|u_T^h(n, x, a) - u_{T,\rho}^h(n, x, a)| \leq L_u \int_{B(\rho)} \|y\| \beta_\rho(y) dy \leq L_u \rho.$$

□

Note 3.2. The functions $u_T^h, u_{T,\rho}^h$ and the operator $A^{h,b}$ verify the inequalities

$$u_T^h(n, x, a) \leq (A^{h,b} u_T^h(n+1, \cdot, \cdot))(x, a) \quad \forall b \in I_h(a), \quad (3.8)$$

$$u_{T,\rho}^h(n, x, a) \leq ((A^{h,b} u_T^h(n+1, \cdot, \cdot))(\cdot, a) * \beta_\rho)(x) \quad \forall b \in I_h(a), \quad (3.9)$$

where

$$(A^{h,b} u_T^h(n+1, \cdot, \cdot))(x, a) = (1 - \lambda h) u_T^h(n+1, x + hg(x, a), b) + hf(x, a).$$

Proposition ii. Every Lipschitz continuous function $w \in W^{1,\infty}(\Omega)$ verifies (being L_w its Lipschitz constant)

$$|(A^{h,b}w) * \beta_\rho - A^{h,b}(w * \beta_\rho)| \leq (1 - \lambda h)L_w L_g \rho h + L_f \rho h. \quad (3.10)$$

Besides

$$|(A^{h,b}u_{T,\rho}^h(n, \cdot, \cdot))(x, a) - (A^{h,b}\tilde{u}_{T,\rho}^h(n, \cdot, \cdot))(x, a)| \leq C(1 - \lambda h)L_u \frac{k^2}{\rho}. \quad (3.11)$$

Proof. Let us see first that (3.10) is valid:

$$\begin{aligned} ((A^{h,b}w) * \beta_\rho - A^{h,b}(w * \beta_\rho))(x, a) &= \int_{B(\rho)} ((1 - \lambda h)w(x - \eta + hg(x - \eta, a), b) + hf(x - \eta, a)) \beta_\rho(\eta) d\eta \\ &\quad - (1 - \lambda h) \int_{B(\rho)} w(x - \eta + hg(x, a), b) \beta_\rho(\eta) d\eta - hf(x, a), \end{aligned}$$

then

$$\begin{aligned} &|((A^{h,b}w) * \beta_\rho - A^{h,b}(w * \beta_\rho))| \leq \\ &\leq \int_{B(\rho)} (1 - \lambda h) |w(x - \eta + hg(x - \eta, a), b) - w(x - \eta + hg(x, a), b)| \beta_\rho(\eta) d\eta \\ &\quad + h \int_{B(\rho)} |f(x - \eta, a) - f(x, a)| \beta_\rho(\eta) d\eta \leq (1 - \lambda h)L_w L_g h \rho + L_f h \rho, \end{aligned}$$

inequality that clearly implies (3.10).

The function $u_{T,\rho}^h$ has second derivatives bounded by

$$\|D^2 u_{T,\rho}^h\| \leq \tilde{C} \frac{L_u}{\rho}, \quad (3.12)$$

where \tilde{C} is a constant that only depends on $\beta(\cdot)$, since $u_{T,\rho}^h$ is the regularization of a Lipschitz continuous function. In consequence, from (1.13) and (3.12), the difference between $u_{T,\rho}^h$ and its linear interpolate $\tilde{u}_{T,\rho}^h$ is bounded by $CL_u \frac{k^2}{\rho}$.

$$\begin{aligned} |(A^{h,b}u_{T,\rho}^h(n, \cdot, \cdot))(x, a) - (A^{h,b}\tilde{u}_{T,\rho}^h(n, \cdot, \cdot))(x, a)| &\leq (1 - \lambda h) |(u_{T,\rho}^h - \tilde{u}_{T,\rho}^h)(n, x + hg(x, a), b)| \leq \\ &\leq C(1 - \lambda h)L_u \frac{k^2}{\rho} \end{aligned}$$

and (3.11) is now proved. \square

Proposition iii. The following estimate holds

$$\tilde{u}_{T,\rho}^h(n, x, a) - u_{k,T}^h(n, x, a) \leq \frac{1 - \lambda h}{\lambda} L_u L_g \rho + \frac{L_f}{\lambda} \rho + \frac{1 - \lambda h}{\lambda h} C_1 \frac{L_u k^2}{\rho}. \quad (3.13)$$

Proof. Let

$$E_n = \sup_{x \in \Omega_k, a \in I_h} (\tilde{u}_{T,\rho}^h(n, x, a) - u_{k,T}^h(n, x, a)) = \max_{i=1, \dots, N, a \in I_h} (\tilde{u}_{T,\rho}^h(n, x^i, a) - u_{k,T}^h(n, x^i, a)). \quad (3.14)$$

From the properties (3.9) and (3.10), we have $\forall b \in I_h(a)$

$$u_{T,\rho}^h(n - 1, x, a) = ((A^h u_T^h(n, \cdot, a)) * \beta_\rho)(x) \leq ((A^{h,b} u_T^h(n, \cdot, a)) * \beta_\rho)(x) \leq$$

$$\leq (A^{h,b}(u_T^h(n, \cdot, a) * \beta_\rho))(x) + \phi(x, h, \rho) = (A^{h,b}u_{T,\rho}^h(n, \cdot, a))(x) + \phi(x, h, \rho), \quad (3.15)$$

with

$$|\phi(x, h, \rho)| \leq (1 - \lambda h)L_u L_g h \rho + L_f h \rho.$$

From (1.15), if $w \in W^k$, we have

$$(A_k^h w)(x, a) = \min_{b \in I_h(a)} \{(1 - \lambda h)w(x + hg(x, a), b) + hf(x, a)\} = (A^h \tilde{w})(x, a) \leq (A^{h,b} \tilde{w})(x, a). \quad (3.16)$$

From (3.11), (3.15) and (3.16) we have

$$u_{T,\rho}^h(n-1, x^i, a) \leq (A^{h,b} \tilde{u}_{T,\rho}^h(n, x, a))(x^i) + \phi(x^i, h, \rho) + \varphi(x^i, h, \rho, k),$$

then, $\forall b \in I_h(a)$, the following is valid

$$u_{T,\rho}^h(n-1, x^i, a) \leq A^{h,b} \tilde{u}_{T,\rho}^h(n, \cdot, a) + \phi(x^i, h, \rho) + \varphi(x^i, h, \rho, k)$$

where

$$|\varphi(x, h, \rho, k)| \leq C(1 - \lambda h) \frac{k^2}{\rho}$$

and

$$u_{k,T}^h(n-1, x^i, a) = (A_k^h u_k^h)(n, x^i, a) = (A^{h,\bar{a}} u_{k,T}^h)(n, x^i, a),$$

where \bar{a} makes the minimum of (3.16) for $u_{k,T}^h$, then

$$u_{T,\rho}^h(n-1, x^i, a) - u_{k,T}^h(n-1, x^i, a) \leq (A^{h,\bar{a}} \tilde{u}_{T,\rho}^h)(n, x^i, a) - (A^{h,\bar{a}} u_{k,T}^h)(n, x^i, a) + \phi(x^i, h, \rho) + \varphi(x^i, h, \rho, k).$$

From (3.7), we have

$$\tilde{u}_{T,\rho}^h(n-1, x^i, a) - u_{k,T}^h(n-1, x^i, a) \leq (1 - \lambda h) (\tilde{u}_{t,\rho}^h(n, x^i, a) - u_{k,T}^h(n, x^i, a)) + \phi(x^i, h, \rho) + \varphi(x^i, h, \rho, k).$$

In consequence, taking into account the definition (3.14), we have

$$E_{n-1} \leq (1 - \lambda h)E_n + (1 - \lambda h)L_u L_g h \rho + L_f h \rho + C(1 - \lambda h)L_u \frac{k^2}{\rho} \quad (3.17)$$

Finally, since

$$E_{n-1} \leq \delta E_n + B \text{ implies } E_0 \leq \delta^n E_n + \frac{B}{1 - \delta},$$

replacing this estimate in (3.17) and taking into account that $E_\mu = 0$, we obtain (3.13), valid for every $0 \leq n \leq \mu$, $x \in V_k$, $a \in I_h$. \square

Proposition iv. *The following estimate holds*

$$u_{k,T}^h(n, x, a) - \tilde{u}_{T,\rho}^h(n, x, a) \leq \frac{1 - \lambda h}{\lambda} L_u L_g \rho + \frac{L_f}{\lambda} \rho + \frac{1 - \lambda h}{\lambda h} C_1 \frac{L_u k^2}{\rho} + 2L_u \rho. \quad (3.18)$$

The proof of this proposition is completely similar to the previous one and we shall omit it. We are now in position to prove Lemma 3.2.

Proof of Lemma 3.2

Proof. For each $0 \leq n \leq \mu$, we will show

$$\max_{0 \leq n \leq \mu} |u_T^h(n, x, a) - u_{k,T}^h(n, x, a)| \leq M\phi(T) \frac{k}{\sqrt{h}}.$$

We start by decomposing the following way:

$$\begin{aligned} |u_T^h(n, x, a) - u_{k,T}^h(n, x, a)| &\leq |u_T^h(n, x, a) - u_{T,\rho}^h(n, x, a)| + |u_{T,\rho}^h(n, x, a) - \tilde{u}_{T,\rho}^h(n, x, a)| \\ &\quad |\tilde{u}_{T,\rho}^h(n, x, a) - u_{k,T}^h(n, x, a)|, \end{aligned} \quad (3.19)$$

applying the previous propositions, (3.13) and (3.18), we can estimate the terms of the right side of (3.19) and obtain

$$|u_T^h(n, x, a) - u_{k,T}^h(n, x, a)| \leq L_u \rho + C \frac{L_u k^2}{\rho} + \frac{1}{\lambda} L_u L_g \rho + \frac{L_f}{\lambda} \rho + \frac{1}{\lambda h} C_1 \frac{L_u k^2}{\rho} + 2L_u \rho. \quad (3.20)$$

Analysis of the different cases

- $L_g > \lambda$.

In this case, L_u has the form

$$L_u = L_f \frac{1}{L_g - \lambda} e^{(L_g - \lambda)T}$$

and the inequality (3.20) becomes

$$|u_T^h(n, x, a) - u_{k,T}^h(n, x, a)| \leq M e^{(L_g - \lambda)T} \left(\rho + \frac{k^2}{h\rho} \right),$$

where

$$M = \max \left(3 + \frac{L_g}{\lambda} + \frac{L_g - \lambda}{\lambda}, C + \frac{C_1}{\lambda} \right) \frac{L_f}{L_g - \lambda}.$$

Minimizing in ρ the expression $(\rho - \frac{k^2}{h\rho})$, we have that the minimum of (3.19) is attained when $\rho = \frac{k}{\sqrt{h}}$, then

$$|u_T^h(n, x, a) - u_{k,T}^h(n, x, a)| \leq 2M e^{(L_g - \lambda)T} \frac{k}{\sqrt{h}}.$$

- $L_g < \lambda$.

In this case $L_u = \frac{L_f}{\lambda - L_g}$ and the inequality (3.18) becomes

$$|u_T^h(n, x, a) - u_{k,T}^h(n, x, a)| \leq M \left(\rho + \frac{k^2}{h\rho} \right),$$

where

$$M = \max \left(3 + \frac{L_g}{\lambda} + \lambda - L_g, \frac{C}{\lambda} + \frac{C_1}{\lambda} \right) \frac{L_f}{\lambda - L_g}.$$

Finally, in the same way as in the last case, we get

$$|u_T^h(n, x, a) - u_{k,T}^h(n, x, a)| \leq 2M \frac{k}{\sqrt{h}}.$$

- $L_g = \lambda$.

In this case $L_u = L_f T$ and the inequality (3.18) becomes

$$|u_T^h(n, x, a) - u_{k,T}^h(n, x, a)| \leq MT \left(\rho + \frac{k^2}{h\rho} \right),$$

where

$$M = \max(4, C + \frac{C_1}{\lambda}) L_f.$$

Then

$$|u_T^h(n, x, a) - u_{k,T}^h(n, x, a)| \leq 2MT \frac{k}{\sqrt{h}}.$$

□

4 Proof of Theorem 3.1

Having proved this two Lemmas we proceed now to prove the Theorem 3.1.

Proof. We will use mainly the estimates given by the previous lemmas. From (2.6), (3.3) and (3.5) we have

$$\begin{aligned} |u(x, a) - u_k^h(x, a)| &\leq |u(x, a) - u_T(0, x, a)| + |u_T(0, x, a) - u_T^h(0, x, a)| + \\ &\quad + \max_n |u_T^h(n, x, a) - u_{k,T}^h(n, x, a)| + |u_{k,T}^h(0, x, a) - u_k^h(x, a)| \\ &\leq \frac{M_f}{\lambda} e^{-\lambda T} + Ch e^{(L_g - \lambda)T} + M\phi(T) \frac{k}{\sqrt{h}} + \frac{M_f}{\lambda} e^{-\lambda T}. \end{aligned} \quad (4.1)$$

By virtue of the lemmas 2.1, 3.1, 3.2, the analysis of (3.2) can be divided into the three following cases:

- $L_g > \lambda$.

$$|u(x, a) - u_k^h(x, a)| \leq M_1 \left(e^{-\lambda T} + e^{(L_g - \lambda)T} \left(h + \frac{k}{\sqrt{h}} \right) \right), \quad (4.2)$$

where

$$M_1 = \max \left(2 \frac{M_f}{\lambda}, C, M \right).$$

Minimizing (4.2) in the T variable, we obtain

$$|u(x, a) - u_k^h(x, a)| \leq K \left(h + \frac{k}{\sqrt{h}} \right)^\gamma \quad (4.3)$$

where

$$\gamma = \frac{\lambda}{L_g}, \quad K = M_1 \left(\left(\frac{1 - \gamma}{\gamma} \right)^\gamma + \left(\frac{1 - \gamma}{\gamma} \right)^{\gamma - 1} \right).$$

The minimum in (4.3) refers to the instance where the minimum is taken in a point of the discrete set $\{nh/n = 0, 1, \dots\}$; for the general case, in (4.3), we must add to the obtained bound a term of order h , which does not modify essentially the estimate.

- $L_g < \lambda$.

In this case, by virtue of (4.1) we have

$$|u(x, a) - u_k(x, a)| \leq \frac{M_f}{\lambda} e^{-\lambda T} + Ch + M \frac{k}{\sqrt{h}} + \frac{M_f}{\lambda} e^{-\lambda T};$$

so, by taking limit $T \rightarrow +\infty$, we obtain

$$|u(x, a) - u_k^h(x, a)| \leq M_1 \left(h + \frac{k}{\sqrt{h}} \right), \quad (4.4)$$

where $M_1 = \max(C, M)$.

- $L_g = \lambda$.

In this case, by virtue of (3.5), (4.1), we have

$$|u(x, a) - u_k^h(x, a)| \leq \frac{M_f}{\lambda} e^{-\lambda T} + ChT + M \frac{k}{\sqrt{h}} T + \frac{M_f}{\lambda} e^{-\lambda T}$$

then

$$|u(x, a) - u_k^h(x, a)| \leq M_1 \left(e^{-\lambda T} + \left(h + \frac{k}{\sqrt{h}} \right) T \right) \quad (4.5)$$

where

$$M_1 = \max \left(2 \frac{M_f}{\lambda}, C, M \right).$$

For $\frac{1}{\lambda} \left(h + \frac{k}{\sqrt{h}} \right) \leq 1$, the minimum in the right side of (4.5) is assumed by

$$\bar{T} = -\frac{1}{\lambda} \log \frac{1}{\lambda} \left(h + \frac{k}{\sqrt{h}} \right);$$

then, replacing in (4.5) we obtain:

$$|u(x, a) - u_k^h(x, a)| \leq M_1 \left(h + \frac{1}{\lambda} \left(h + \frac{k}{\sqrt{h}} \right) - \frac{1}{\lambda} \left(h + \frac{k}{\sqrt{h}} \right) \log \left(\frac{1}{\lambda} \left(h + \frac{k}{\sqrt{h}} \right) \right) \right).$$

Since the following property holds

$$-x \log(x) \leq K x^\gamma, \quad \gamma \in (0, 1), \quad K = \frac{1}{1-\gamma} e^{-1},$$

we have

$$|u(x, a) - u_k^h(x, a)| \leq C \left(h + \frac{k}{\sqrt{h}} \right)^\gamma,$$

where $C = \frac{M_1}{\lambda} \max(1, \frac{1}{1-\lambda} e^{-1})$.

□

4.1 About the choice of the discretization parameters

Taking into account the central result (3.1) we can reach the following conclusions:

- When h is of order k , that is $c_1k \leq h \leq c_2k$ being c_1 and c_2 constants, the convergence is of order $k^{\frac{\gamma}{2}}$, that is

$$|u(x, a) - u_k^h(x, a)| \leq Ck^{\frac{\gamma}{2}}.$$

- The best speed of convergence is attained selecting h of order $k^{\frac{2}{3}}$, that is $c_1k^{\frac{2}{3}} \leq h \leq c_2k^{\frac{2}{3}}$. In this case we obtain the estimate

$$|u(x, a) - u_k^h(x, a)| \leq Ck^{\frac{2}{3}\gamma}. \quad (4.6)$$

- The estimate (4.6) is optimal, in the sense that it is possible to construct examples where

$$|u(x, a) - u_k^h(x, a)| \geq ck^{\frac{2}{3}\gamma}$$

The construction of these examples can be made by following the techniques used in [11].

5 Example

We consider a simple example in \mathbb{R}^2 . The controlled dynamics is:

$$(D) \begin{cases} \dot{y}(s) = g(y(s), \alpha(s)) & s > 0 \\ y(0) = x = (x_1, x_2) \end{cases} \quad (5.1)$$

where

$$g(x_1, x_2, a) = (-(a+1)x_1, -(a+1)x_2)$$

and $\mathcal{A}(a)$ is defined as in the general case and $\Omega = (-1, 1)^2$. The cost to minimize is given by

$$u(x, a) = \inf_{\alpha \in \mathcal{A}(a)} \int_0^\infty f(y(s), \alpha(s)) e^{-s} ds.$$

Here

$$f(x_1, x_2, a) := a \left(\frac{1}{4} - (x_1^2 + x_2^2) \right).$$

It is clear that g and f verify the general hypotheses. Here, the discount factor is $\lambda = 1$.

We present now a full discretization of the problem. We introduce the discretization parameter h to define a equispaced discretization in the time and control spaces. The control variable a takes values in the set

$$I_h := \left\{ ih \mid i = 0 \dots \frac{1}{h} \right\}.$$

We also define $I_h(a) = I_h \cap [a, 1]$. We proceed now to define the discretization of the state space. For this aim for each $k > 0$ we introduce a family of finite elements in the following way: first we define the set of vertices

$$V_k := \{P_{i,j}(-1 + ik, -1 + jk); i, j = 1, \dots, 2\frac{1}{k} + 1\} \quad (5.2)$$

Note 5.1. for the sake of simplicity we assume here that $\frac{1}{k}$ is an integer.

These vertices define the family of triangles $S_{i,j}^k$ whose union over i, j is

$$\Omega_k = [-1 + k, 1 - k]^2$$

Let us check that this family of triangulations verify the hypotheses. Being diam $S_{i,j}^k = k$ for all j it is clear that (1.10) is verified. The functions g^1, g^2 are specially defined so that (1.11) holds in both cases. From the definition of Ω_k , (1.12) is straightforward. Finally we observe that taking $\chi_1 = r = 1$, (1.13) and (1.14) are verified.

The space W_k is the set of piecewise linear continuous functions defined in $\Omega_k \times I_h$. We recall that the fully discrete solution u_k^h is the unique solution of the fixed point problem of the operator A_k^h . Let $u_0 \in W_k$ be the zero function. We will consider it as the initial function in our fixed point problem and for every $n \in \mathbb{N}$ we note

$$u_n = A_k^h(u_{n-1}).$$

We have the following estimate

$$\|u_n - u_k^h\| \leq \frac{1}{h} \|u_n - u_{n-1}\| \quad (5.3)$$

For different choices of h, k we obtain approximations to the solution u_k^h by applying repeatedly the operator A_k^h to the initial function $u_0 \in W_k$. Actually in the implementations that will be shown here we take always $h = k$. In each case the stop criterion is

$$\|u_n - u_{n-1}\| \leq h^2$$

since from (5.3) this estimate assures us that

$$\|u_n - u_k^h\| \leq h.$$

In the following table we show the numerical results.

	(D)	
h=k	Iterations	Time(secs)
0.50	1	0.0168
0.40	1	0.0025
0.30	2	0.0120
0.20	3	0.0995
0.10	10	3.5145
0.05	33	135.8481
0.02	126	13527.0552

The algorithm was implemented in *Scilab 5.4.1 - 64 bits* in a *PC* equipped with an *Intel Core i7-3770K* processor with *8 GB DD3 RAM*.

6 Conclusions

In this work mainly we have developed efficient procedures of numerical resolutions of the Hamilton-Jacobi-Bellman equation associated to the optimal control problem with monotone controls, making use of the particular structure of the original problem to develop the discretization schemes and obtain its estimates of the speed of convergence.

Essentially, the discretization employed is equivalent to the resolution of a finite family of concatenated stopping time problems. According to this point of view, it can be obtained by applying mechanically the results contained in [5, 11], an estimate of the speed of convergence of order $k^{\frac{7}{4}}$. This estimate is improved in this work analyzing directly the control problem rather than taking this way leading to non optimal bounds.

Finally, we can remark that the optimization of the discretization parameters has been studied and it has been established that in general the choice of a relationship $h \approx k^{\frac{2}{3}}$ allows us to get the best possible result.

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