Quasi-exact solvability and entropies of the one dimensional regularised Calogero model

Quasi-exact solvability and entropies of the one dimensional regularised Calogero model

Federico M. Pont,[1,*] Omar Osenda,[1,†] and Pablo Serra[1,‡]

[1] *Facultad de Matemática, Astronomía y Física, Universidad Nacional de Córdoba and IFEG-CONICET, Ciudad Universitaria, X5016LAE Córdoba, Argentina*

(Dated: February 15, 2018)

Abstract

The Calogero model can be regularised through the introduction of a cutoff parameter which removes the divergence in the interaction term. In this work we show that the one-dimensional two-particle regularised Calogero model is quasi-exactly solvable and that for certain values of the Hamiltonian parameters the eigenfunctions can be written in terms of Heun's confluent polynomials. These eigenfunctions are such that the reduced density matrix of the two-particle density operator can be obtained exactly as well as its entanglement spectrum. We found that the number of non-zero eigenvalues of the reduced density matrix is finite in these cases. The limits for the cutoff distance going to zero (Calogero) and infinity are analysed and all the previously obtained results for the Calogero model are reproduced. Once the exact eigenfunctions are obtained, the exact von Neumann and Rényi entanglement entropies are studied to characterise the physical traits of the model. The quasi-exactly solvable character of the model is assessed studying the numerically calculated Rényi entropy and entanglement spectrum for the whole parameter space.

* pont@famaf.unc.edu.ar
† osenda@famaf.unc.edu.ar
‡ serra@famaf.unc.edu.ar

I. INTRODUCTION

The Calogero model [1] occupies a remarkable place in theoretical and mathematical physics. It has been linked, to give a few examples, to advances made in quantum Hall effect [2], random matrices [3], integrability [4] and Yang-Mills theory [5].

Remarkably, the Calogero model variants, the deformed [6, 7], the different generalisations [8, 9] and the regularised ones [10], inherit many of its properties, a trend that was acknowledged from very early by Sutherland [11]. More recently, it has been shown that the $p-$reduced density matrix (p-RDM) of a N-particle one-dimensional Calogero model can also be obtained exactly [12, 13], as well as the entanglement spectrum, for a discrete set of the strength interaction parameter (let us recall that the p-RDM matrix is obtained when $(N-p)$ particles are traced out of the density matrix of an $N-$particle system). Besides, at these values, the Rényi entanglement entropies show non-analytical behaviour [14] in contradistinction with the von Neumann entropy.

The harmonic confinement potential, present in all the variants of the model, was included more as a mean to keep the particles bounded, since the interaction between them is mainly repulsive, rather than as a model for an implementable potential. Many other models, also share the confinement property with the Calogero model, to name a few, the so-called spherium [16–18], or other electron systems confined in boxes with different geometries, as square [19, 20], cylindrical [21] and spherical [22]. These models present quasi-exact solvability [15], which means that the spectrum and the eigenfunction are exactly known in a discrete set of the Hamiltonian parameters. Besides, its behaviour is quite different from the one observed in extended systems, so they posed new challenges to the application of numerical methods, such as the DFT method [23, 24] or accurate variational expansions [34]. Recently, there has been a flurry of activity in the quasi-exact solvability subject [10, 18, 25–27], while early examples can be found in the works of Kais *et. al.* [28] and Taut [29]. The recent advances made in one- and two-particle models with quasi-exact solvability rely heavily on the properties of the polynomial solutions of the Heun differential equation [30].

The broad application of the Heun's equation and its polynomial solutions to many different problems, in classical and quantum physics, has been made possible by the work of Fiziev [31]. In particular it has been applied to the study of the dynamics of a rotor vibratory gyroscope [32], the calculation of natural occupation numbers in two electron quantum-

rings [25], the solution of the Schrödinger equation for a particle trapped in a hyperbolic double-well potential [26], the problem of two electrons confined on a hypersphere [18], one electron in crossed inhomogeneous magnetic and homogeneous electric fields [27], and in the study of normal modes in non-rotating black holes [33].

Quite recently, Downing has reported some analytical solutions of the three-dimensional two-particle regularised Calogero model[10]. The regularisation is done introducing a short-distance cutoff parameter d, which prevents the divergence of the potential when the distance between the particles goes to zero. Remarkably, the model is quasi-exactly solvable so, for a given value of d, the exact two-particle wave function can be obtained only for a discrete set of values of the interaction strength parameter g, as is usually denoted in the context of the Calogero model. At these values, the two-particle wave function is a polynomial function of the inter-particle distance. As has been shown in Ref. [12], when the eigenfunctions of a multi-particle Calogero model can be written as the product of a polynomial function depending on the inter-particle distances, times a function that depends separately on the coordinates of each particle, then the p-RDM and the entanglement spectrum can be both obtained exactly. Since the Rényi entropy also shows a very particular behaviour for these discrete set of values, it does beg the question of how many of these features are inherited by the regularised model. To this end, we study the exact solutions of the one-dimensional two particle regularised model, both their symmetric and anti-symmetric solutions under particle interchange and their reduced density matrices. Even though this is a quasi-exactly solvable model we calculate numerical solutions to study the whole parameter space of the Hamiltonian.

This paper is organised as follows: The regularised Calogero model is presented in Section II. In Section III the exact symmetric wave functions are thoroughly analysed while the antisymmetric ones are the subject of Section IV. The von Neumann and Rényi entropies for the exact two-particle states, together with numerical approximations, are presented in Section V. Finally, a discussion of the results and some open questions are presented in Section VI.

II. THE MODEL AND ITS EIGENFUNCTIONS

Recently, Downing [10] showed that the three-dimensional two-particle regularised Calogero model is solvable for a discrete set of values of the interacting parameter. In this work, we address the one dimensional two-particle regularised Calogero Hamiltonian

$$H = h(1) + h(2) + \frac{g}{x_{12}^2 + 2\,d^2}\,, \tag{1}$$

where

$$h(i) = \frac{1}{2}\,p_i^2 + \frac{1}{2}\,x_i^2 \;\text{ and }\; x_{12} = |x_1 - x_2|\,. \tag{2}$$

In particular, we look for a discrete set of exact two-particle symmetric or antisymmetric wave functions. We do not assume particular values for the spin variable, so the symmetric and antisymmetric functions can be used to construct two-fermions or two-bosons solutions depending on the symmetry of the spinorial part of the quantum state.

With the coordinate transformation

$$X = \frac{1}{\sqrt{2}}(x_1 + x_2) \;\;;\;\; x = \frac{1}{\sqrt{2}}(x_1 - x_2)\,, \tag{3}$$

the Hamiltonian Eq. (1) takes the form $H = H_X + H_x$, where

$$H_X = -\frac{1}{2}\frac{d^2}{dX^2} + \frac{1}{2}X^2 \quad ; \tag{4a}$$

$$H_x = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2 + \frac{g/2}{x^2 + d^2}\,. \tag{4b}$$

The eigenfunctions will be the product of eigenfunctions of each Hamiltonian,

$$\psi(x_1, x_2) = \Psi(X)\,\psi(x), \tag{5}$$

and the eigenenergies the sum of the eigenvalues, $E = E_X + E_x$. For the center of mass Hamiltonian Eq.(4a) we will consider the ground state

$$E_X = \frac{1}{2} \;\;;\;\; \Psi(X) = \frac{1}{\pi^{1/4}} e^{-X^2/2}\,. \tag{6}$$

This eigenfunction is symmetric, and the Hamiltonian Eq. (4b) is even in x, that means that the even (odd) eigenfunctions of the Hamiltonian Eq. (4b) correspond to totally symmetric (antisymmetric) eigenfunctions under particle interchange. The odd eigenfunctions

of Hamiltonian Eq. (4b) are the three-dimensional solutions written by Downing in Ref. [10] for zero angular momentum times x (see Eq. (8) below).

Following [10], in order to find the eigenfunctions of the relative Hamiltonian Eq. (4b), we perform two consecutive changes of variables,

$$z = \left(\frac{x}{d}\right)^2 \;\; ; \;\; \psi(z) = e^{-d^2 z/2} y(z) \quad \text{and} \quad \xi = -z \;\; ; \;\; y(\xi) = (1-\xi) f(\xi) \tag{7}$$

for the symmetric eigenfunctions, and

$$z = \left(\frac{x}{d}\right)^2 \;\; ; \;\; \psi(z) = e^{-d^2 z/2} \sqrt{z}\, y(z) \quad \text{and} \quad \xi = -z \;\; ; \;\; y(\xi) = (1-\xi) f(\xi) \tag{8}$$

for the antisymmetric ones. The function $f(\xi)$ fulfills the standard form of the confluent Heun differential equation [31],

$$f'' + \left(\alpha + \frac{\beta+1}{\xi} + \frac{\gamma+1}{\xi-1}\right) f' + \left(\frac{\mu}{\xi} + \frac{\nu}{\xi-1}\right) f = 0\,, \tag{9}$$

where the parameters are defined as

$$\alpha = d^2 \; ; \; \beta = \mp\frac{1}{2} \; ; \; \gamma = 1 \; ; \; \mu = \frac{1}{4}\left(d^2(1-k^2) + g - 2\right) \; ; \; \nu = d^2 - \frac{g-2}{4}\,. \tag{10}$$

and $k^2 = 2E_x$. The difference between one-dimensional symmetric or antisymmetric functions is given by the coefficient $\beta = -1/2$ and $\beta = 1/2$, respectively.

As usual (see [31]), we define the parameters

$$\eta = \frac{1}{2}(\alpha - \beta - \gamma + \alpha\beta - \beta\gamma) - \mu = \frac{1}{4}(d^2k^2 - g + 2) \;\; ; \;\; \delta = \nu - \eta - \frac{1}{2}(\alpha + \beta + \gamma + \alpha\gamma + \beta\gamma) = -\frac{d^2k^2}{4}\,, \tag{11}$$

and the confluent Heun function is written as

$$f(\xi) = \sum_{m=0}^{\infty} v_m(\alpha, \beta, \gamma, \delta, \eta)\, \xi^m\,, \tag{12}$$

where the coefficients are given by the recurrence relation

$$A_m v_m = B_m v_{m-1} + C_m v_{m-2} \;\; ; \;\; v_0 = 1 \; ; \; v_{-1} = 0 \tag{13}$$

where

$$A_m = 1 + \frac{\beta}{m}\,, \tag{14a}$$

$$B_m = 1 + \frac{-\alpha+\beta+\gamma-1}{m} + \frac{\eta-(-\alpha+\beta+\gamma)/2-\alpha\beta/2+\beta\gamma/2}{m^2}\,, \tag{14b}$$

$$C_m = \frac{\alpha}{m^2}\left(\frac{\delta}{\alpha} + \frac{\beta+\gamma}{2} + m - 1\right)\,. \tag{14c}$$

We note that the parameters α, γ, δ and η and the recurrence relations are those of the three dimensional bosonic case for zero angular momentum [10].

The confluent Heun functions are not square-integrable [30] and the series must be truncated in order to obtain a polynomial of degree N in Eq.(12), which implies, from Eq. (13), $v_{N+1} = v_{N+2} = 0$. Therefore the condition

$$C_{m=N+2} = 0 \tag{15}$$

in Eq. (14c) gives the eigenenergies

$$k_N^2 \;=\; 2E_x \;=\; 4N + 6 \mp 1 \;\Rightarrow\; E_N = E_x + E_X \;=\; 2N + \frac{7 \mp 1}{2}\,, \tag{16}$$

where the upper (lower) sign describes symmetric (antisymmetric) states.

Note that the energies E_N of the polynomial solutions are independent of g and d. Since there is no indication that the eigenfunctions of the Hamiltonian Eq. (4b) should have the same energy for any couple of values of the parameters g and d, the polynomial solutions with energy E_N must be restricted to *isoenergetic* curves $g^{(N)}(d)$ in the (d, g)-plane. Moreover, the energy E_N must then match the energy of one of the polynomial solutions of the Calogero model for $d = 0$. The complete spectrum of the Calogero model for any g is given by

$$E_n = n + \frac{1}{2}\sqrt{1+4g} + \frac{3}{2}, \tag{17}$$

where n is the principal quantum number [1]. The polynomial solutions of the Calogero model are defined by an index p [12] that corresponds to the parametrization

$$g_p \;=\; p(p-1)\;, p = 2, 3, 4 \dots\;, \tag{18}$$

and the spectrum is given by

$$E_{n;p} = n + p + 1\;;\; n = 0, 2, 4 \dots\;;\; p = 2, 3, 4 \dots\;. \tag{19}$$

The regularised Calogero model has a denumerable infinite number of exact polynomial solutions for a fixed value of d. Each one of them corresponds (in the limit $d \to 0$) to one of the polynomial solutions of the Calogero model. This correspondence allows to label the isoenergetic curves $g^{(N)}(d)$ with the index p (Eq. 18) by the relation

$$g_p^{(N)}(d=0) = g_p. \tag{20}$$

In other words, the spectrum of the polynomial solutions of the regularised Calogero model has been labelled with two different sets of indices , E_N (Eq. (16)) and $E_{n;p}$ (Eqs. (19) and (20)). So, E_N and $E_{n;p}$ must satisfy the relation

$$E_N \; = \; E_{n;p} \Rightarrow n + p = 2N + \frac{5 \mp 1}{2}. \tag{21}$$

Note that N and p are not quantum numbers, so we will obtain ground- and excited-state eigenfunctions for different values of N and p. It is straightforward to show, using the Hellman-Feynman theorem, that the curves $g_p^{(N)}(d)$ are monotonically increasing functions of d. So, the number of exact polynomial solutions of the regularised model for a fixed value of g is given by the number of polynomial solutions of the Calogero model which fulfil $g_p < g$, see Fig. 4a.

III. SYMMETRIC EIGENFUNCTIONS WITH $N = 0$ AND $N = 1$

The expressions of the polynomial eigenfunctions for a fixed value of N and a given set of parameters can be computed using Eqs. (12)-(16). The corresponding reduced density matrices are finite and can be obtained following the procedure described in Ref. [12]. For a better understanding it is useful to write down the simplest cases $N = 0$ and $N = 1$.

A. N=0

In this case, Eq.(14c) gives $k_N^2 = 5$, or $E_N = 3$. The condition $v_1 = 0$ gives the isoenergetic curve

$$g = 2 + 4d^2\,, \tag{22}$$

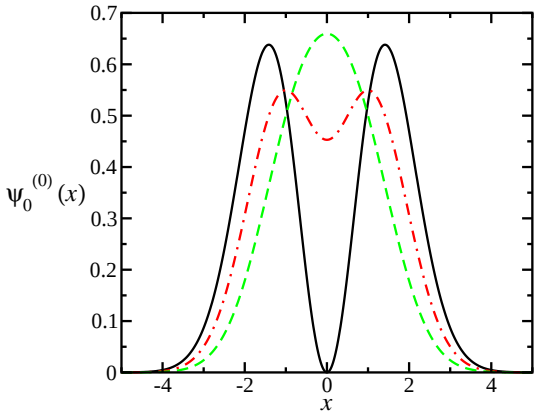

FIG. 1. Symmetric ground state wave function for $N = 0$, $\psi^{(0)}_{0,2}(d)$, for three different values of the cutoff length $d = 0, 1, 2$ (full black, dash-dotted red and dashed green lines respectively).

so that $g(d = 0) = 2$ and, according to Eq. (20), $p = 2$. Hence we recast g in Eq. (22) as $g_2^{(0)}(d)$. From Eq. (21), the only compatible solution is $n = 0$, which corresponds to a ground state. The wave function for the reduced coordinate is given by

$$\psi^{(0)}_{0,2}(x) = \frac{2(d^2 + x^2)}{\pi^{1/4}\sqrt{3 + 4d^2 + 4d^4}} e^{-x^2/2}, \tag{23}$$

where the subscripts and superscripts are chosen according to the prescription $\psi^{(N)}_{n,p}(x)$. It is interesting to note that $\psi^{(0)}_{0,2}(0)$ is a minimum (maximum) for $d < \sqrt{2}$ ($d > \sqrt{2}$ respectively). This phenomenon is shown in Fig. 1, where the wave function Eq. (23) is plotted for $d = 0, d = 1$ and $d = 2$.

For a given two-particle state $\psi(x_1, x_2)$, the 1-RDM is defined as

$$\rho(x, y) = \int dz\, \psi^*(x, z)\psi(y, z), \tag{24}$$

then, replacing Eqs. (23) and (6) in Eq.(5), we obtain

$$\rho^{(0)}_{0,2}(x, y) = \frac{e^{-(x^2+y^2)/2}}{4\sqrt{\pi}\,(3 + 4d^2 + 4d^4)}(3 + 8d^2 + 16d^4 + 2(1 + 4d^2)(x^2 + y^2) + 8xy + 4x^2y^2)\,. \tag{25}$$

Using the orthonormal Hermite functions

$$\psi_k(x) = \frac{e^{-\frac{1}{2}x^2} H_k(x)}{\sqrt{2^k k!\pi^{1/2}}}, \tag{26}$$

where $H_k(x)$ are the Hermite polynomials, the 1-RDM can be written as

$$\rho^{(0)}_{0,2}(x;y) \;=\; \sum_{i,j=0}^{2} \rho_{i,j}(d)\,\psi_i(x)\,\psi_j(y)\,. \tag{27}$$

The 1-RDM above can be cast in matrix form

$$\left[\rho^{(0)}_{0,2}\right](d) \;=\; \begin{pmatrix} \frac{3/8+d^2+d^4}{3/4+d^2+d^4} & 0 & \frac{1+2d^2}{4\sqrt{2}(3/4+d^2+d^4)} \\ 0 & \frac{1}{4(3/4+d^2+d^4)} & 0 \\ \frac{1+2d^2}{4\sqrt{2}(3/4+d^2+d^4)} & 0 & \frac{1}{8(3/4+d^2+d^4)} \end{pmatrix}, \tag{28}$$

and its eigenvalues can be exactly calculated and are given by

$$\lambda_{\pm} = \frac{2+4d^2+4d^4 \pm (1+2d^2)\sqrt{3+4d^2+4d^4}}{2(3+4d^2+4d^4)}, \tag{29a}$$

$$\lambda_o = \frac{1}{4(3/4+d^2+d^4)}\,. \tag{29b}$$

These eigenvalues are showed in Fig. 2. In the important limit $d \to 0$ we obtain

$$\left[\rho^{(0)}_{0,2}\right](0) \;=\; \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{3\sqrt{2}} \\ 0 & \frac{1}{3} & 0 \\ \frac{1}{3\sqrt{2}} & 0 & \frac{1}{6} \end{pmatrix}, \tag{30}$$

whose eigenvalues are

$$\lambda_{\pm} \;=\; \frac{2\pm\sqrt{3}}{6}\,, \quad \lambda_o \;=\; \frac{1}{3}\,, \tag{31}$$

as reported in Ref. [12]. For $d \to \infty$, replacing $g_2^{(0)}(d)$ in Eq. (1), the interaction potential becomes a constant, and the 1-RDM correspond to two non-interacting harmonic particles in the ground state,

$$\lim_{d\to\infty}\left[\rho^{(0)}_{0,2}\right](d) \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{32}$$

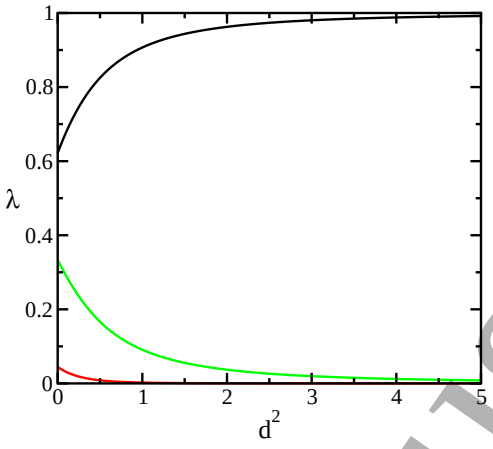

FIG. 2. Eigenvalues of the $(N = 0)$-reduced density matrix for the symmetric solution, λ_+ (black line), λ_- (red line) and λ_0 (green line) (Eq. (29)), as a function of the squared cutoff length d^2.

B. N=1

In this case $k_N^2 = 9$, which implies $E_N = 5$. The value for v_1 is

$$v_1(d, g) = 1 + 4d^2 - \frac{g}{2}, \tag{33}$$

and the condition $v_2 = 0$ gives two isoenergetic curves

$$g_\pm = 7 + 6d^2 \pm \sqrt{25 - 12d^2 + 4d^4}. \tag{34}$$

Performing the same analysis for the functions $g_\pm(d)$ as was done previously for $g_2^{(0)}(d)$, we obtain $g_+(d = 0) = 12 \Rightarrow p = 4 \Rightarrow g_+(d) = g_4^{(1)}(d)$, and similarly $g_-(d) = g_2^{(1)}(d)$. Then, Eq. (21) gives two solutions for the energy, $E_{0,4}$, corresponding to the ground state for $g = 12$, and $E_{2,2}$ corresponding to the second excited state for $g = 2$. $g_4^{(1)}$ corresponds to the ground state, with the nodeless wave function

$$\psi^{(1)}_{0,4}(x) = \frac{2\sqrt{2}\left(2d^4 + d^2(7 - 2d^2 + D)x^2 + (5 - 2d^2 + D)x^4\right)e^{-x^2/2}}{\pi^{1/4}\sqrt{525(5+D) + 2d^2\left(60 + 75D + 2d^2\left(2d^2(8 + 6d^2 - D) - 3(8 + D)\right)\right)}}, \quad (35)$$

where $D = \sqrt{25 + 4(d^2 - 3)d^2}$. Note that $\psi^{(1)}_{0,4}(x)$ is a Gaussian times a fourth degree polynomial, then the 1-RDM is a 5×5 matrix. The limiting cases of this wave function for $d \to 0$ and $d \to \infty$ are

$$\psi^{(1)}_{0,4}(x)\Big|_{d=0} = \frac{4x^4 e^{-x^2/2}}{\pi^{1/4}\sqrt{105}} \quad \text{and} \quad \psi^{(1)}_{0,4}(x)\Big|_{d\to\infty} = \frac{e^{-x^2/2}}{\pi^{1/4}}\,, \quad (36)$$

respectively. The former wave function correspond to the Calogero ground state for $g = 12$, and the latter one to the ground state of two non-interacting particles in an harmonic potential.

Taking $g^{(1)}_2$ we obtain the second excited state, whose wave function is

$$\psi^{(1)}_{2,2}(x) = \frac{2\sqrt{2}\left(2d^4 + d^2(-7 + 2d^2 + D)x^2 + (5 - 2d^2 - D)x^4\right)e^{-x^2/2}}{\pi^{1/4}\sqrt{-525(-5+D) + 2d^2\left(60 - 75D + 2d^2\left(2d^2(8 + 6d^2 + D) + 3(-8 + D)\right)\right)}}\,, \quad (37)$$

which has two nodes. In the limit $d \to 0$, the wave function takes the form

$$\psi^{(1)}_{2,2}(x)\Big|_{d=0} = \frac{\sqrt{2}}{\sqrt{15}\pi^{1/4}} x^2(2x^2 - 5)e^{-x^2/2}\,, \quad (38)$$

which corresponds to the second excited state for the Calogero model with $g = 2$, and for the limit $d \to \infty$ is

$$\psi^{(1)}_{2,2}(x)\Big|_{d\to\infty} = \frac{H_2(x)e^{-x^2/2}}{2\sqrt{2}\pi^{1/4}}\,. \quad (39)$$

In this limit, the complete wave function is given by

$$\psi(x_1, x_2)|_{d\to\infty} = \Psi(X)\ \psi^{(1)}_{2,2}(x)\Big|_{d\to\infty} = \frac{1}{2}\left(\psi_2(x_1)\,\psi_0(x_2) + \psi_0(x_1)\,\psi_2(x_2)\right) - \frac{1}{\sqrt{2}}\psi_1(x_1)\,\psi_1(x_2)\,. \quad (40)$$

For two non-interacting harmonic oscillators the only three products of one-particle eigenfunctions with total energy $E = 5$, are exactly those appearing in Eq. (40). Interestingly, the probability to find the two-particle system in a product of even one-particle eigenfunctions is equal to the probability to find it in a product of odd ones.

IV. ANTISYMMETRIC EIGENFUNCTIONS WITH N=0

The energy in this case is $k_N^2 = 7$ or $E_N = 4$, and the condition $v_1 = 0$ gives

$$g = 6 + 4d^2 = g_3^{(0)}(d)\,, \tag{41}$$

then, it corresponds to the antisymmetric ground state $E_{0,3}$. The reduced wave function is given by

$$\psi_{0,3}^{(0)}(x) \;=\; \frac{2\sqrt{2}x(d^2+x^2)}{\pi^{1/4}\sqrt{15+12d^2+4d^4}}e^{-x^2/2}\,, \tag{42}$$

and the 1-RDM takes the form

$$\begin{aligned}\rho_{0,3}^{(0)}(x,y) \;=\; \frac{e^{-(x^2+y^2)/2}}{\pi^{1/2}8\,(15+12d^2+4d^4)} \quad & \big(15+24d^2+16d^4+(54+48d^2+32d^4)xy \\ & +6(3+4d^2)(x^2+y^2)+4(3+4d^2)xy(x^2+y^2) \\ & +36x^2y^2+8x^3y^3\big)\,.\end{aligned} \tag{43}$$

The corresponding matrix in the Hermite basis set, Eq. (26), is

$$\left[\rho_{0,3}^{(0)}\right](d) \;=\; \frac{1}{4(15+12d^2+4d^4)}\begin{pmatrix} 21+24d^2+8d^4 & 0 & 3\sqrt{2}(3+2d^2) & 0 \\ 0 & 27+24d^2+8d^4 & 0 & \sqrt{6}(3+2d^2) \\ 3\sqrt{2}(3+2d^2) & 0 & 9 & 0 \\ 0 & \sqrt{6}(3+2d^2) & 0 & 3 \end{pmatrix}, \tag{44}$$

and its eigenvalues, that are shown in Fig. 3, are

$$\lambda_\pm \;=\; \frac{1}{4}\left[1 \pm \frac{\sqrt{2(99+4d^2(3+d^2)(15+2d^2(3+d^2))}}{(15+2d^2(3+d^2))}\right]\,, \tag{45}$$

both with multiplicity 2. In the limit $d \to 0$ we obtain the expressions reported for the two-fermion Calogero model [12]

$$\left[\rho_{0,3}^{(0)}\right](0) \;=\; \begin{pmatrix} \frac{7}{20} & 0 & \frac{3}{10\sqrt{2}} & 0 \\ 0 & \frac{9}{20} & 0 & \frac{1}{10}\sqrt{\frac{3}{2}} \\ \frac{3}{10\sqrt{2}} & 0 & \frac{3}{20} & 0 \\ 0 & \frac{1}{10}\sqrt{\frac{3}{2}} & 0 & \frac{1}{20} \end{pmatrix}, \tag{46}$$

whose eigenvalues are

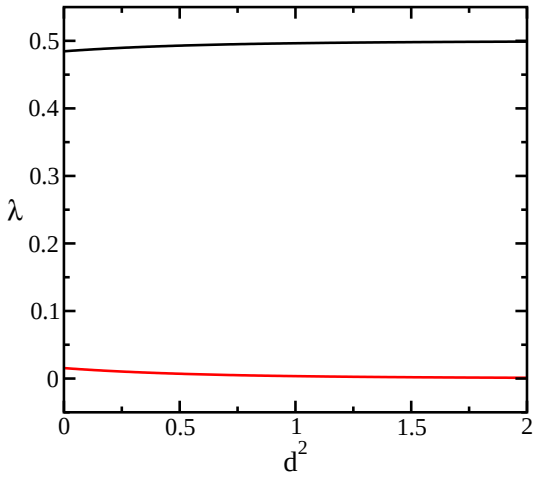

FIG. 3. Eigenvalues of the $(N = 0)$-reduced density matrix for the antisymmetric solution, λ_+ (black line), λ_- (red line) (Eq. (43)), as a function of the squared cutoff length d^2. Note that both eigenvalues are doubly degenerate.

$$\lambda_\pm = \frac{5 \pm \sqrt{22}}{20} . \tag{47}$$

For the limit $d \to \infty$ we get

$$\lim_{d\to\infty} \left[\rho^{(0)}_{0,3}\right](d) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \tag{48}$$

and

$$\psi(x_1, x_2)|_{d\to\infty} = \Psi(X)\ \psi^{(0)}_{0,3}(x)\Big|_{d\to\infty} = \frac{1}{\sqrt{2}} \left(\psi_0(x_1)\,\psi_1(x_2) - \psi_1(x_1)\,\psi_0(x_2)\right) , \tag{49}$$

which is the antisymmetric state with the lowest energy for a two non-interacting particles system in an harmonic potential.

V. THE VON NEUMANN AND RÉNYI ENTROPIES

So far, we have only analysed the quasi-exact solvability property of the regularised Calogero model. This Section is devoted to analyse the behaviour of the entanglement

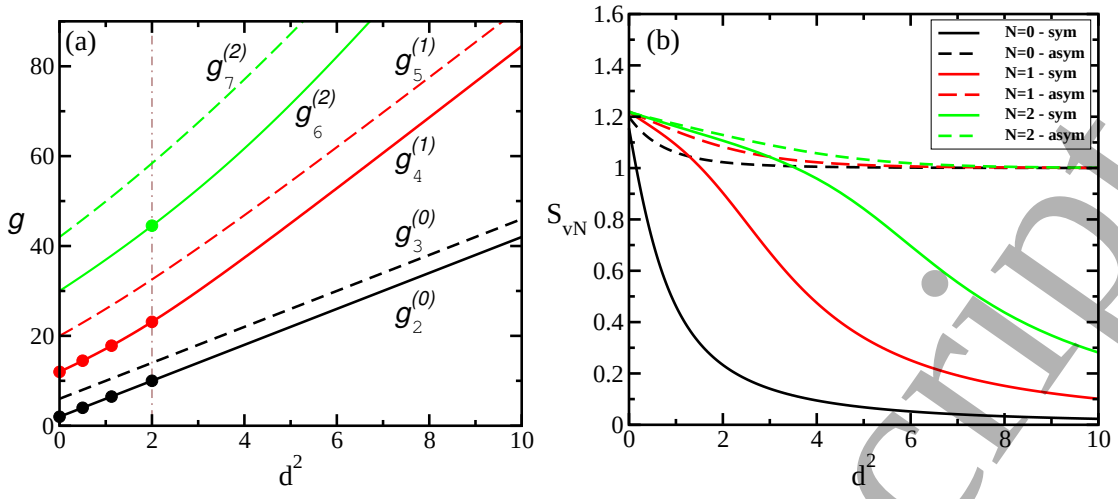

FIG. 4. (a) Isoenergetic curves $g_p^{(N)}(d)$ where an exact polynomial solution of the one-dimensional regularised Calogero model is known. From bottom to top the solid (dashed) lines correspond to symmetric (antisymmetric) ground-state wave functions for $N = 0, 1, 2$, respectively. The vertical brown dash-dotted line corresponds to $d^2 = 2$. The dots over this line are those shown at the bottom of Fig. 5(a) and pinpoint the values of g for which a number of eigenvalues of the approximate 1-RDM become null, see the text. The dots over the curves $g_2^{(0)}(d)$ and $g_4^{(1)}(d)$ correspond to those shown at the bottom of Fig. 6. (b) Exact von Neumann entropy of the ground-state wave functions over the isoenergetic curves $g_p^{(N)}(d)$ shown in (a). The same colour code is used in both panels.

entropies for both the exact polynomial solutions and arbitrary pairs of the pair (g, d).

For a given density matrix ρ with an entanglement spectrum $\{\lambda_i\}$, its spectral decomposition is

$$\rho = \sum_i \lambda_i \, |\phi_i\rangle \, \langle\phi_i| \, , \tag{50}$$

where the $|\phi_i\rangle$'s are the eigenvectors or natural orbitals of ρ. The eigenvalues $\{\lambda_i\}$ are also known as natural occupation numbers.

The Rényi entropy of ρ is defined as

$$S^a(\rho) = \frac{1}{1-a} \log_2 \mathrm{Tr}\rho^a \, , \quad a \neq 1 \, , \tag{51}$$

where $a > 0$ (here we use a instead of the more common α parameter to avoid possible

conflicts with the parameter α in Eq. (10)). Besides, it is well known that

$$\lim_{a\to 1} S^a(\rho) = S_{vN}(\rho) = -\mathrm{Tr}(\rho \log_2 \rho), \tag{52}$$

where $S_{vN}(\rho)$ is the von Neumann entropy. In some cases, the mono-parametric family of Rényi entropies can be used to shed light over the peculiarities of the entanglement spectrum, *i.e.* the spectrum of the density matrix under study, because of its ability to weight differently the eigenvalues of ρ by changing the value of a. This is made clear by looking at the expressions of both entropies, Eqs. (51) and (52), in terms of the eigenvalues of ρ

$$S^a(\rho) = \frac{1}{1-a}\log_2\left(\sum_i \lambda_i^a\right), \qquad S_{vN} = -\sum_i \lambda_i \log_2 \lambda_i. \tag{53}$$

Let us start by inspecting the isoenergetic curves $g_p^{(N)}(d)$ in the (d, g)-plane. Figure 4(a) show these curves for all the ground states $n = 0$. Note that the curves hit the ordinate axis at the g values where an exact polynomial solution of the Calogero model can be found [12]. The von Neumann entropies (vNE) along each isoenergetic curve are presented in Fig. 4(b). The vNE for the symmetric case goes to zero for large values of the cutoff length parameter indicating a single natural orbital population. Conversely, in the antisymmetric case the vNE converges to a limiting value of one because the antisymmetrization prevents such single natural orbital population. Note that the vNE at $d = 0$ is not the same for all curves and whether symmetric or antisymmetric configurations have larger vNE depends upon the particular $g_p^{(N)}(0)$ value as shown in [12] for the Calogero model.

We turn now to the study of the 1-RDM eigenvalues and vNE for arbitrary values of the parameters in the (d, g)-plane. The eigenvalues were calculated using a high precision variational method with a symmetrical Hermite-DVR basis set function [35, 36], so the eigenvalues correspond to a symmetric problem. The method to obtain the eigenvalues of the 1-RDM from the approximate two-particle variational wave function has been discussed elsewhere [37–39]. Figure 5(a) shows the typical behaviour of the largest eigenvalues of the 1-RDM for a given value of d as a function of g. There is a discrete set of values of g where an infinite number of eigenvalues become null. In the following we will show that this set of values lies over the isoenergetic curves $g_p^{(N)}(d)$.

The values of g where a number of eigenvalues become null for a given fixed value of d can be plotted in the (d, g)-plane. Figure 4(a) shows these values for $d^2 = 2$ as filled dots and

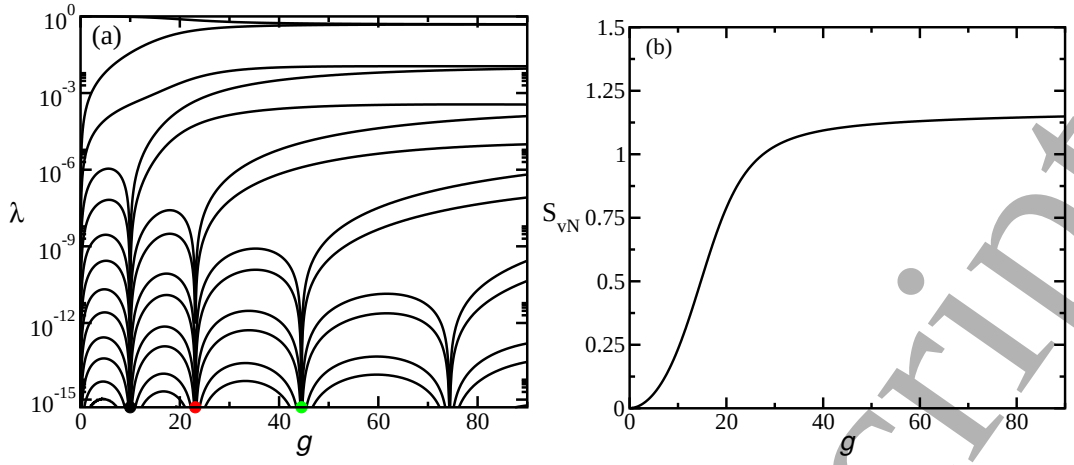

FIG. 5. (a) Largest approximate eigenvalues of the 1-RDM as a function of g. The 1-RDM was obtained from the variational wave function approximation to the symmetric ground state wave function for $d^2 = 2$. The coloured dots at the bottom indicate the g values for which a number of eigenvalues become null, and are also indicated in Fig. 4(a). (b) Approximate von Neumann entropy corresponding to the eigenvalues shown in panel (a).

they match those shown at the bottom of Fig. 5(a). It is clear that the values of g where a number of eigenvalues become null coincide with those found in the previous Sections, since the curves shown in Fig. 4(a) correspond to the analytical equations found for the lowest eigenvalues corresponding to symmetric and antisymmetric functions, see Eqs. (22),(34) y (41). Besides, the number of non-zero natural occupation numbers over each curve is always the same, and coincides with the number found in Sections III and IV. From bottom to top in Fig. 4(a) the number is equal to three, five, and so on, for the symmetric eigenvalues. The same can be said for the eigenvalues corresponding to the antisymmetric eigenfunctions.

Since the isoenergetic curves $g_p^{(N)} = g_p^{(N)}(d)$ are increasing functions of d^2, it is clear that the values of g where a number of eigenvalues become null are also increasing functions of d. This can be appreciated in Figure 6 where the seventh eigenvalue of the 1-RDM of the symmetric two-particle wave function is shown for several values of d. The sixth and seventh eigenvalues are the largest eigenvalues that have only two zeros. If λ_i is the i-th eigenvalue of the 1-RDM, and g_n^i is the n-th value of g such that $\lambda_i(g_n^i) = 0$, then $g_n^i < g_{n+1}^i$ and $g_n^i(d_1) < g_n^i(d_2)$, $\forall\ d_1 < d_2$.

The Rényi entropies also provide a tool to identify where the number of non-vanishing

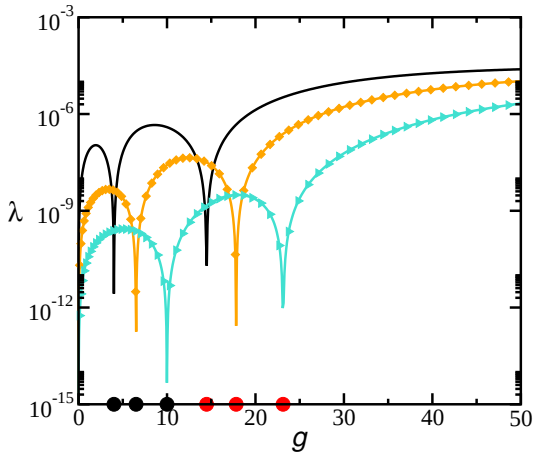

FIG. 6. The seventh largest eigenvalue of the 1-RDM constructed from the variational approximation to the symmetric ground state wave function. The curves shown correspond to three values of the squared cutoff length $d^2 = 0.5, 1.125, 2$ (black, red and green, respectively). The dots on the abscissa axis indicate the values of g for which the eigenvalue shows a behaviour compatible with an almost vanishing quantity. These points are the same as those shown in Fig. 4(a) over the isoenergetic curves $g_2^{(0)}(d)$ and $g_4^{(1)}(d)$ for symmetric solutions.

eigenvalues of the RDM alternates between a finite value and infinity. For the Calogero model it has been shown that the entanglement spectrum has a numerable infinite number of non-zero elements in open sets of the interaction parameter. These open sets are separated from each other by a discrete set of values of the interaction parameter, $\bar{g}_n$, where the number of non-zero eigenvalues of the entanglement spectrum is finite [14]. As has been shown above for the regularised Calogero model, the set of values of the parameter where the entanglement spectrum is finite depends on the actual value of d, which implies that $\bar{g}_n$ is a function of d.

The eigenvalues of the 1-RDM for fixed values of d are analytical functions of g. This fact allows us to assume a concrete analytical expression for the eigenvalues. As a consequence, explicit expressions for the Rényi entropies and its derivatives can be written. We develop here the case for symmetric two-particle wave function (the anti-symmetric case is similar), where the 1-RDM has only $2n + 1$ non-zero eigenvalues at $g = \bar{g}_n$, in the following the dependency with d is dropped to keep the notation as simple as possible.

The following results will only rely on the analyticity of the eigenvalues around isolated points in the parameter space where the spectrum is finite. Assuming that

$$\lambda_i(g) \sim \begin{cases} \lambda_i(\bar{g}_n) + \lambda_i^{(1)}(g-\bar{g}_n) & \text{if } \; i \leq 2n+1 \\ \\ \lambda_i^{(2)}(g-\bar{g}_n)^{2k_{i,n}} & \text{if } i > 2n+1 \end{cases} \quad \text{for } g \to \bar{g}_n \,, \tag{54}$$

where $\lambda_i^{(1)}, \lambda_i^{(2)}$ are constants, and $k_{i,n} \geq 1$ is an integer. Eq. (53) can be written as

$$\begin{aligned} S^a(g) &= \frac{1}{1-a} \log_2 \left(\sum_{i=1}^{2n+1} \lambda_i^a(g) + \sum_{i=2n+2}^{\infty} \lambda_i^a(g) \right) \\ &= \frac{1}{1-a} \left(\log_2 \left(\sum_{i=1}^{2n+1} \lambda_i^a(g) \right) + \log_2 \left(1 + \frac{\sum_{i=2n+2}^{\infty} \lambda_i^a(g)}{\sum_{i=1}^{2n+1} \lambda_i^a(g)} \right) \right) \\ &\underset{g \to \bar{g}_n}{\sim} \frac{1}{1-a} \left(\log_2 \left(\sum_{i=1}^{2n+1} \lambda_i^a(g) \right) + \frac{\sum_{i=2n+2}^{\infty} \lambda_i^a(g)}{\ln 2 \sum_{i=1}^{2n+1} \lambda_i^a(g)} \right) = S_n^a(g) + s_n^a(g) \,. \end{aligned} \tag{55}$$

The last equality defines the quantities $S_n^a(g)$ and $s_n^a(g)$. So, it is clear that $S_n^a(\bar{g}_n) = S^a(\bar{g}_n)$, and $s_n^a(\bar{g}_n) = 0$. Then, the derivative of the Rényi entropy at $g = \bar{g}_n$ can be obtained as

$$\begin{aligned} \left. \frac{\partial S^a(g)}{\partial g} \right|_{g=\bar{g}_n} = \left. \frac{\partial S_n^a(g)}{\partial g} \right|_{g=\bar{g}_n} + \\ \frac{a}{\ln 2 \, (1-a)} \left(\frac{\sum_{i=2n+2}^{\infty} \lambda_i^{a-1}(g) \partial_g \lambda_i(g)}{\sum_{i=1}^{2n+1} \lambda_i^a(g)} - \frac{\sum_{i=2n+2}^{\infty} \lambda_i^a(g) \sum_{i=1}^{2n+1} \lambda_i^{a-1}(g) \partial_g \lambda_i(g)}{\left(\sum_{i=1}^{2n+1} \lambda_i^a(g) \right)^2} \right)_{g=\bar{g}_n} \end{aligned} \tag{56}$$

The first term in Eq. (56) is a well-defined constant and the third one is zero. As a result, the analytical properties of the derivative are determined by the second term. Using the analytic expansion of the eigenvalues, Eq. (54), and assuming that k_m is the minimum value of $k_{i,n}$, the leading asymptotic behavior of s_n^a is

$$s_n^a(g) \underset{g \to \bar{g}_n}{\sim} C_n \left((g-\bar{g}_n)^{2k_m} \right)^a = C_n \left| g - \bar{g}_n \right|^{\chi k_m} , \tag{57}$$

where $\chi = 2a$, which implies that

$$\frac{\partial s_n^a(g)}{\partial g} \underset{g \to \bar{g}_n}{\sim} \chi k_m C_n \left| g - \bar{g}_n \right|^{\chi k_m - 1} sign(g - \bar{g}_n) \,. \tag{58}$$

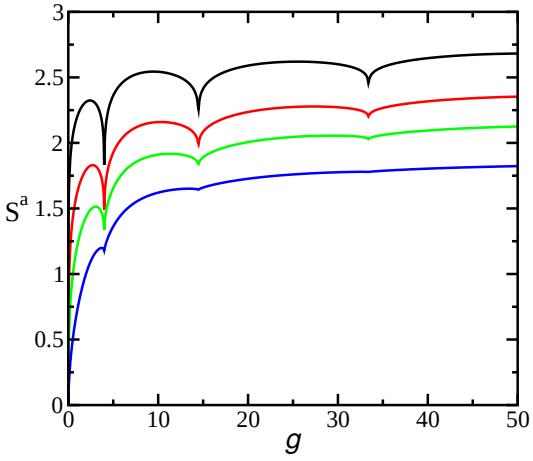

FIG. 7. Rényi entropies as a function of the interaction parameter g and a cutoff length parameter $d^2 = 0.5$. The shown curves correspond to the parameter values $a = 0.1, 0.15, 0.2, 0.3$ (black, red, green and blue line respectively). The kinks in all the curves are located at values of g for which there is an exact finite expansion of the 1-RDM, which renders a finite Hilbert. As discussed in the text, the kinks are a consequence of the analytic properties of the vanishing eigenvalues at those values of g.

Collecting the results of Eqs. (54) to (58), the derivative of the Rényi entropy can be expressed as

$$\left.\frac{\partial S^a(g)}{\partial g}\right|_{g=\bar{g}_n} = \begin{cases} \left.\begin{array}{ll} -sign(C_n)\times\infty & \text{for} \quad g\to\bar{g}_n^- \\ sign(C_n)\times\infty & \text{for} \quad g\to\bar{g}_n^+ \end{array}\right\} & \text{if} \;\; \chi k_m < 1 \\ \left.\begin{array}{ll} \partial_g S_n^a(\bar{g}_n) - C & \text{for} \quad g\to\bar{g}_n^- \\ \partial_g S_n^a(\bar{g}_n) + C & \text{for} \quad g\to\bar{g}_n^+ \end{array}\right\} & \text{if} \;\; \chi k_m = 1 \\ \partial_g S_n^a(\bar{g}_n) & \text{if} \;\; \chi k_m \geq 1\,. \end{cases} \tag{59}$$

Even tough the derivative of S^a is continuous for $\chi \geq 1$, it is straightforward to see from the eigenvalues asymptotic behaviour, Eq. (54), that the second derivative diverges for $1 < \chi k_m < 2$, but it is analytical for $\chi k_m = 2$, *i.e* the kink at $\chi k_m = 1$ is smoothed until it disappears at $\chi k_m = 2$.

Figure 7 shows the behaviour of S^a as a function of g for different values of the parameter a at $d^2 = 0.5$. The kinks at fixed values of g can be easily appreciated, as well as their softening for increasing values of a as predicted by Eq. (59). Observe that the bottom curve

corresponds to the largest value of a depicted, while the upper curve corresponds to the smallest one. Keeping d fixed ensures that the interaction values $\bar{g}_n$ (where only a finite number of eigenvalues are non-zero) are also kept fixed and, as a consequence, the kinks in the curves calculated for different values of a are located at the same abscissas.

VI. DISCUSSION

Models with quasi-exact solvability have wave functions that are polynomial functions on the inter-particle distance so, at least for those that do not depend on any angular variable but the ones on the inter-particle radius, they should also possess exact and finite reduced density matrices. This last problem is open for three dimensional problems with non-trivial angular momentum.

For the model analysed in this work, the quasi-exact solvability character is intertwined with the fact that the Calogero model has exact solutions that can be expressed as polynomials in the interparticle distance. So, when we take the limit $d \to 0$ *over* the isoenergetic curves we are able to recover all the quantities corresponding to the Calogero model. Then, it is natural to wonder if a given model that has quasi-exact solvability, also has an exactly solvable limiting model.

Recently, there has been a number of works dealing with the properties of the entanglement spectrum, or natural occupation numbers, in particular the phenomenon of *pinning* [40–42]. The pinning is related to the the generalised Pauli constraints (GPC) which are a set of (in)equalities that generalise the Pauli exclusion principle. These constraints are defined trough affine inequalities that confine the values of the 1-RDM eigenvalues to D-dimensional polytopes, where D is the dimension of the 1-particle Hilbert space. Pinning, or quasi-pinning, of the 1-RDM eigenvalues of a solution refers to the near-saturation of such GPC's. Much of the understanding has been obtained analysing systems of coupled harmonic oscillators (Moshinsky model), because they are amenable to a complete analytical treatment. The nearly exclusive use of harmonic oscillator models is not surprising since models with exact solutions are scarce. Even more scarce are models which also have exact and finite reduced density matrices, as the one presented in this work together with the Calogero model. In this sense, Calogero and regularised Calogero models provide exact solutions and 1-RDM eigenvalues which may help the efforts made to understand the pinning

or quasi-pinning phenomenon (in principle, for any dimension D).

Our results shown that the Rényi entropy is a capable tool to identify systems with exact and finite RDM. Nevertheless, to improve its usability it is necessary to determine if a set of very small eigenvalues are effectively zero or not. To accomplish this it is necessary to identify if, for example, performing a finite size analysis of the numerical eigenvalues at the parameter where the system has an exact and finite RDM the behaviour is (quite) different from the behaviour where there is not such a RDM. It is clear that for models with wave functions with only a polynomial dependency on the inter-particle distance it is possible to choose a finite basis for the Hilbert space where the wave function to be analysed is contained exactly, resulting in an exact RDM. In this case, the RDM derived from the finite basis contains all the information required to produce a finite number of non-zero eigenvalues and a number of exactly zero ones. Conversely, when the finite basis set used to analyse a given wave function does not contain the exact wave function under consideration there will be a number of eigenvalues that should be zero in the limit of an infinite basis set, but for a finite basis they are not, and a numerical criterion is in order. Work around these lines is in progress.

ACKNOWLEDGMENTS

We acknowledge SECYT-UNC and CONICET (PIP-11220150100327CO) for partial financial support.

[1] F. Calogero, J. Math. Phys. **10**, 2191 (1969).

[2] H. Azuma and S. Iso, Phys. Lett. B **331**, 107 (1994).

[3] B. D. Simons, P. A. Lee, and B. L. Altshuler, Phys. Rev. Lett. **70**, 4122 (1993).

[4] A. P. Polychronakos, Phys. Rev. Lett. **70**, 2329 (1993).

[5] A. Gorsky and N. Nekrasov, Nucl. Phys. B **414**, 213 (1994).

[6] O. Chalykh, M. Feigin, and A. Veselov, J. Math. Phys. **39**, 695 (1998).

[7] F. Atai and E. Langmann, J. Math. Phys. **58**, 011902 (2017).

[8] A. P. Polychronakos, Nucl. Phys. B **543**, 485 (1999).

[9] M. Znojil and M. Tater, J. Phys. A: Math.Gen. **34**, 1793 (2001).

[10] C.A. Downing, Phys. Rev. A **95**, 022105 (2017).

[11] B. Sutherland, J. Math. Phys. **12**, 246 (1971).

[12] O. Osenda, F. Pont, A. Okopińska, and P. Serra, J. Phys. A: Math. Gen. **48**, 485301 (2015).

[13] H.Katsura and Y. Hatsuda, J. Phys. A: Math. Theor. **40**, 13931 (2007).

[14] M. Garagiola, E. Cuestas, F. M. Pont, P. Serra, and O. Osenda, Phys. Rev. A **94**, 042115 (2016).

[15] A.G Ushveridze, *Quasi-Exactly Solvable Models in Quantum Mechanics* (Institute of Physics, Bristol, 1994).

[16] G.S. Ezra and R.S. Berry, Phys. Rev. A **25**, 1513 (1982).

[17] G.S. Ezra and R.S. Berry, Phys. Rev. A **28**, 1989 (1983).

[18] P.-F. Loos and Peter M. W. Gill, Phys. Rev. Lett. **103**, 123008 (2009).

[19] A. Alavi, J. Chem. Phys. **113**, 7735 (2000).

[20] S. Ghosh and P. M. W. Gill, J. Chem. Phys. **122**, 154108 (2005).

[21] I. G. Ryabinkin and V. N. Staroverov, Phys. Rev. A **81**, 032509 (2010).

[22] J. Jung, P. Garcia-Gonzalez, J. E. Alvarellos, and R. W. Godby, Phys. Rev. A **69**, 052501 (2004).

[23] P.-F. Loos, C. J. Ball, and P. M.W. Gill, J. Chem. Phys. **140**, 18A524 (2014).

[24] P.-F. Loos, Phys. Rev. A **89**, 052523 (2014).

[25] V. Tognetti, and P.-F. Loos, J. Chem. Phys. **144**, 054108 (2016).

[26] C. A. Downing, J. Math. Phys. **54**, 072101 (2013).

[27] C. A. Downing, and M. E. Portnoi, Phys. Rev. B **94**, 045430 (2016).

[28] S. Kais, D. R. Herschbach, and R. D. Levine, J. Chem. Phys. **91**, 7791 (1989).

[29] M. Taut, Phys. Rev. A **48**, 3561 (1993).

[30] A. Ronveaux, editor, *Heun's Differential Equations* (Oxford University Press, Oxford, U.K., 1995).

[31] P.P. Fiziev, J. Phys. A: Math. Gen. **43**, 035203 (2010).

[32] K.A. Motsepe, M.Y. Shatalov and S.V. Joubert, Appl. Math. Comput. **239**, 4755 (2014).

[33] P. P. Fiziev and D. R. Staicova, Phys. Rev. D **84**, 127502 (2011).

[34] L. G. Jiao, L. R. Zan, Y. Z. Zhang, and Y. K. Ho, Int. J. Quantum Chem. **117**, e25375 (2017).

[35] U. Manthe, J. of Phys.: Condens. Matt. **29**, 253001 (2017).

[36] M. H. Beck, A. Jäckle, G. A. Worth, and H.-D. Meyer, Phys. Rep. **324**, 1 (2000).

[37] O. Osenda and P. Serra, Phys. Rev. A **75**, 042331 (2007).

[38] O. Osenda, P. Serra and S. Kais, Int. J. Quantum Inf. **6**, 303 (2008).

[39] K. J. H. Giesbertz and R. van Leeuwen, J. Chem. Phys. **139**, 104109 (2013).

[40] C. Schilling, D. Gross and M. Christandl, Phys. Rev. Lett. **110**, 040404 (2013).

[41] C. L. Benavides-Riveros, J. M. Gracia-Bondía and M. Springborg, Phys. Rev. A **88**, 022508 (2013).

[42] R. Chakraborty and D. A. Mazziotti, Int. J. Quantum Chem. **115**, 1305 (2015).